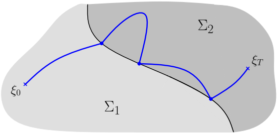
Optimal Control of Hybrid Systems with Regional Dynamics

– Presentation at the Measurement and Control Laboratory, ETH Zürich –

Angela Schöllig

January 30th, 2008



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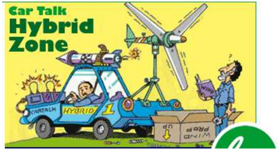

Combining different elements.

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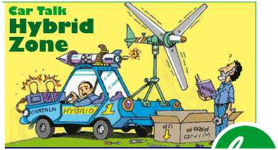

Combining different elements.

Definition: Hybrid System

A dynamic system which exhibits both *continuous and discrete* dynamic behavior.

Hybrid Systems with Regional Dynamics

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A dynamic system, where the governing (continuous) dynamics $\dot{x} = f_i(x,u)$ vary depending on the region D_i, the continuous state x is evolving in.

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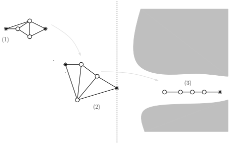
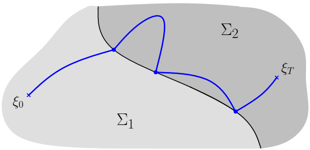
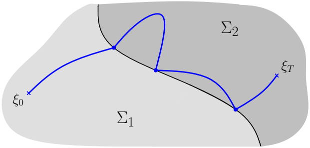


Outline

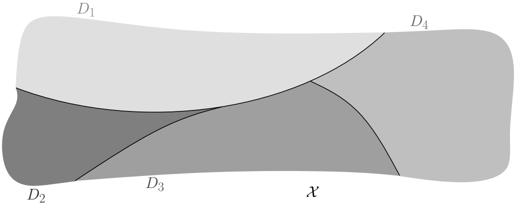


Outline

1. The Multiregional Point-To-Point Problem
2. The Basic Idea
3. A Dynamic Programming Approach
4. Conclusions



Regions and Geometric Framework



- state space $\mathcal{X} \subset \mathbb{R}^n$:

$$\mathcal{X} = \bigcup_{i=1}^{q} (D_i \cup \partial D_i), \quad D_i \cap D_j = \emptyset,\ i \neq j$$

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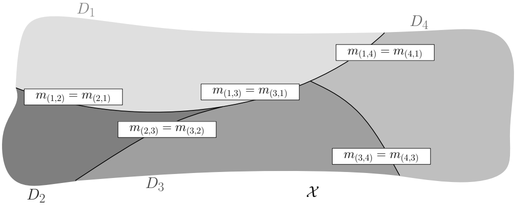

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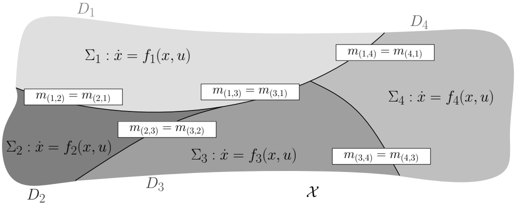

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- switching manifolds $m_{(i,j)}$:

$$m_{(i,j)} = \partial D_i \cap \partial D_j$$

- continuous dynamics Σ_i:

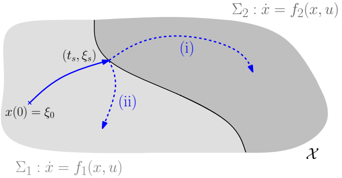
$$\dot{x}(t) = f_i(x(t), u(t)) \quad \text{if} \ \ x(t) \in D_i$$

Transition Behavior

If $\xi_s = \lim_{t \to t_s} x(t) \in m_{(i,j)}$, there are two possibilities of further execution:

(i) "passing through" the switching manifold

(ii) "bouncing back" into the original region



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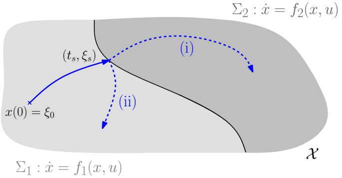
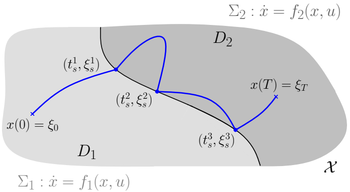


- *discrete* control input $e \in \mathcal{E}$ is needed specifying the behavior at a switching point
- mode-transitions triggered by events in the continuous state space

The Hybrid Execution



- continuous-valued state $x(t) \in \mathcal{X}$
- continuous dynamics within D_i

$$\dot{x}(t) = f_i(x(t), u(t))$$

- continuous-valued control signal

$$u(\cdot) \in \mathcal{U}(U, L_\infty([0,T]))$$

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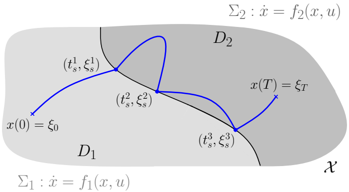

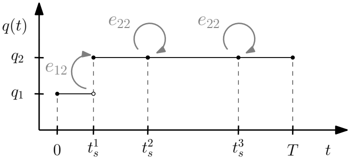

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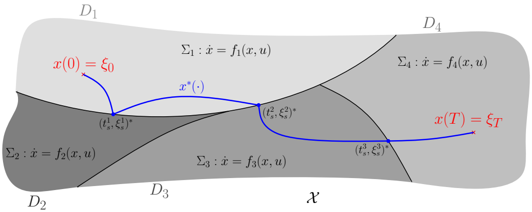
- discrete-valued state $q(t) \in Q$
- discrete dynamics on boundaries

$$q_j = \Gamma\left(q_i, e_{ij}\right)$$

- discrete input sequence

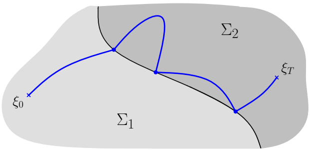
$$w = e_{i_1 j_1} e_{i_2 j_2} e_{i_3 j_3} \dots$$

The Optimization Problem



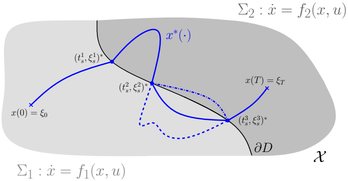
"Given a specific cost function $J = \int_0^T \ell(x(t), u(t))\, dt$, determine the **optimal path** $x^*(\cdot)$ of going from a **given initial state** $x(0) = \xi_0$ to a **fixed final state** $x(T) = \xi_T$ during an *a priori* **specified time horizon** T assuming an upper bound N on the number of transitions along the hybrid trajectory."

Outline



The Principle of Optimality

The Principle of Optimality implicitly states that along an optimal hybrid trajectory $x^*(\cdot)$...



- ... the state's **execution** between $(t_s^m, \xi_s^m)^*$ and $(t_s^{m+1}, \xi_s^{m+1})^*$ is optimal
- ... the part of the optimal hybrid **input** $u^*(\cdot)$ between t_s^m and t_s^{m+1} is optimal
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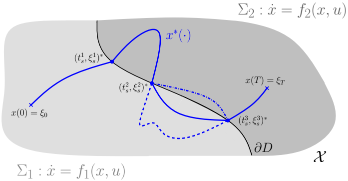

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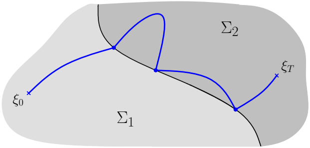
$\Rightarrow$ standard (non-hybrid) state-constrained optimal control problems

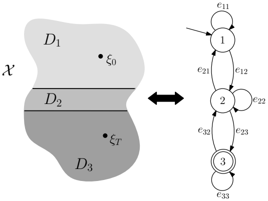
$\Rightarrow$ Dynamic Programming approach

Outline



The Hierarchic Decomposition

(i) Transition Automaton



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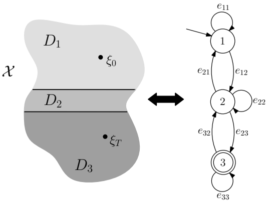

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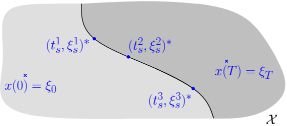

(ii) Hybrid Bellman Equation

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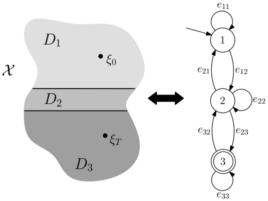
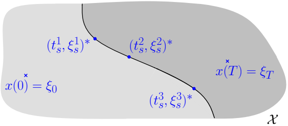
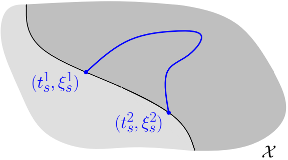




(ii) Hybrid Bellman Equation

(iii) Standard state-constrained optimal control problem



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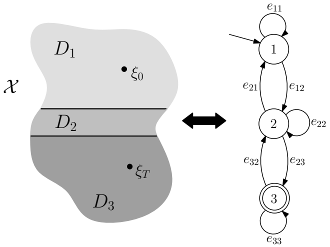

- discrete representation of the geometric structure
- specifies the connections between the individual regions
- contains information about initial and final region D_{i_0}, $\xi_0 \in D_{i_0}$ and D_{i_T}, $\xi_T \in D_{i_T}$

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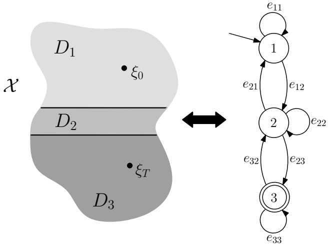

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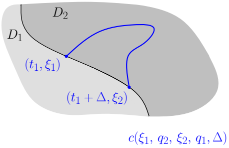
$\Rightarrow$ **associated languages provide *global accessibility relations***

Definitions

Definition (Cost $c(\xi_1, q_{i_1}, \xi_2, q_{i_2}, \Delta)$)

The infimum of the costs associated with driving the system from $\xi_1 \in D_{i_1} \cup \partial D_{i_1}$ to $\xi_2 \in D_{i_1} \cup \partial D_{i_1}$ over a time horizon Δ **without leaving $D_{i_1} \cup \partial D_{i_1}$ and without a switching taking place.**



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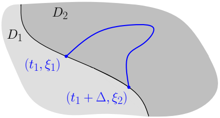
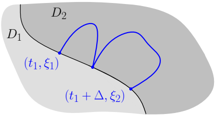


$c(\xi_1, q_2, \xi_2, q_1, \Delta)$



$V^1(\xi_1, q_2, \xi_2, q_1, \Delta)$

Definition (Cost-to-go function $V^M(\xi_1, q_{i_1}, \xi_2, q_{i_2}, \Delta)$)

The infimum of the costs of going from $\xi_1 \in \mathcal{X}$ to $\xi_2 \in \mathcal{X}$ during the time horizon Δ **using exactly M switches and starting in region D_{i_1}.**

The main theorem

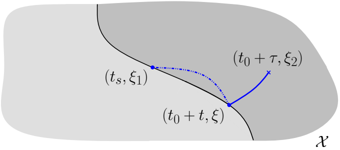
Theorem (The Hybrid Bellman Equation)

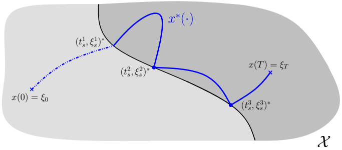
Under the appropriate assumptions and for $0 < K \le M$

$$V^K(\xi_1, q_{i_1}, \xi_2, q_{i_2}, \tau) = \inf_{t\in(0,\tau)} \inf_{\xi\in m_{(i_1,j)}} \inf_{j\in I} \left\{ c(\xi_1, q_{i_1}, \xi, q_j, t) + V^{K-1}(\xi, q_j, \xi_2, q_{i_2}, \tau - t) \right\}$$

such that

$$\begin{aligned} e &= e_{i_1 j}\,, & w &\in F_{K-1}(M,A)\,, \\ \mathit{end}(w) &= i_2\,, & ew &\in F_K(M,A)\,. \end{aligned}$$





Final result

For $K = 0$,

$$V^0(\xi_1, q_{i_1}, \xi_2, q_{i_2}, \tau) = c(\xi_1, q_{i_1}, \xi_2, q_{i_2}, \tau).$$

The optimal cost associated with the original problem

$$W^N(\xi_0, q_{i_0}, \xi_T, q_{i_T}, T) = \min_{0 \leq K \leq N} V^K(\xi_0, q_{i_0}, \xi_T, q_{i_T}, T)$$

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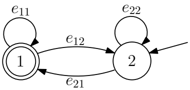
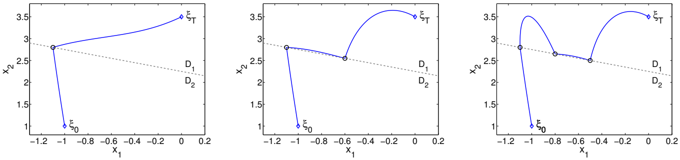
- the costs $c(\cdot,\cdot,\cdot,\cdot,\cdot)$ can be calculated in advance
- the languages associated with the transition automaton constrain the effort needed to accomplish the recursion
- only a discretization of the switching manifolds $m_{(i,j)}$ and the time interval $[0,T]$ is necessary

Bimodal Example (1)

- **Transition Automaton:**



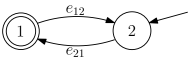
- **Results:**



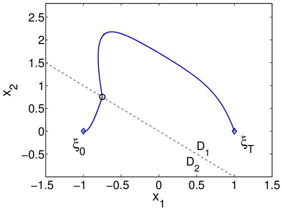
(a) $V^1 \approx 3.865$ (b) $V^2 \approx 3.866$ (c) $V^3 \approx 3.867$

Bimodal Example (2)

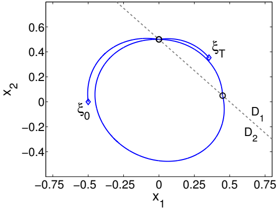
- **Transition Automaton:**



- **Results for two different regional dynamics systems** $(N = 20)$**:**

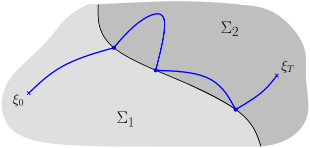


(d) One Switch is Optimal.



(e) Three Switches are Optimal.

Outline



Conclusions

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- Dynamic Programming algorithm:
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  - (ii) numerically implementable calculation rule
- further generalizations possible

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$\Rightarrow$ **global optimality conditions for a very general class of regional dynamics systems**

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