On the Construction of Safe Controllable Regions for Affine Systems with Applications to Robotics

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Motivating Examples

Google Self-Driving Car

RIBA Healthcare Robot

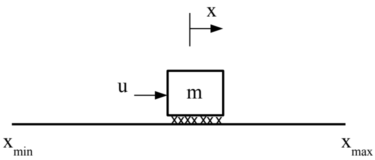
Flying Drones

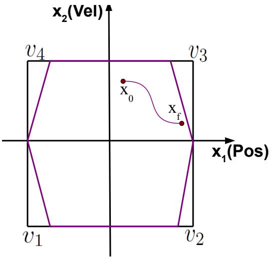
- Safety is critical since these systems interact with humans.

Safety First

Control Design for Safety-Critical Systems

- Urgent need for addressing fundamental questions: When can we fully control a dynamical system under given safety constraints? Kalman's controllability does not apply!

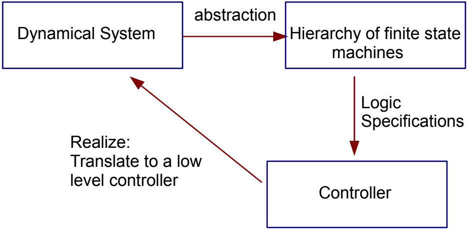




- Introduced in-block controllability (IBC) study

Hierarchical Control of Hybrid/Nonlinear Systems

- ODEs are not powerful for designing controllers satisfying high-level objectives, expressed by temporal logic statements!

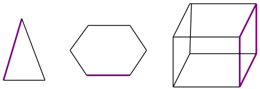


- IBC partitions/covers $\Rightarrow$ Hierarchy of finite state machines
- This talk: Provide constructive guidelines for building IBC partitions/covers

Geometric Background

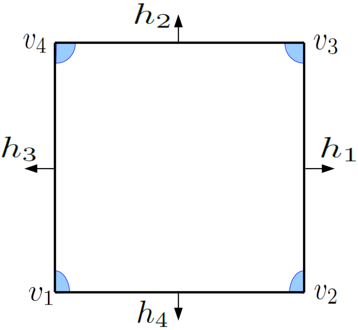
Definition

An *n-dimensional polytope* is the convex hull of a finite set of points in $\mathbb{R}^n$ whose affine hull has dimension n.



- A facet is an $(n-1)$-dimensional face of the polytope.
- An n-dimensional simplex is a special case of an n-dimensional polytope that has $n+1$ vertices.
- A polytope is simplicial if all its facets are simplices.

Notations



$$\mathcal{C}(x) := \{\, y \in \mathbb{R}^n \mid h_j \cdot y \leq 0,\ j \in \{1, \cdots, r\}\ s.t.\ x \in \mathcal{F}_j \}.$$

In-Block Controllability

Definition (In-Block Controllability (IBC))

Consider an affine system $\dot{x}(t) = Ax(t) + Bu(t) + a$ and an n-dimensional polytope X. We say that the affine system is in-block controllable (IBC) w.r.t. X if there exists $M > 0$ such that for all $x,\ y \in X^{\circ}$, there exist $T \geq 0$ and a control input u defined on $[0, T]$ such that (i) $\|u(t)\| \leq M$ and $\phi(x, t, u) \in X^{\circ}$ for all $t \in [0, T]$, and (ii) $\phi(x, T, u) = y$.

- Objective: Provide a computationally efficient method for constructing IBC regions.

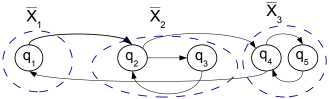
Controlled Invariance Problem

- Controlled Invariance: Find inputs such that all the state trajectories initiated in a set remain in it for all future time.
- IBC vs Controlled Invariance
- Controlled Invariance on given polytopes [GC86], [DH99] $\Rightarrow$ Building controlled invariant polytopic sets [BMM95], [Blan99]
- Analogous to the history of the controlled invariance problem, we extend results for checking IBC on given polytopes to building polytopic regions satisfying the IBC property.

[GC86] Gutman, Cwikel. IEEE Trans. Aut. Control, 1986.
[DH99] Dorea, Hennet. European Journal of Control, 1999.
[BMM95] Blanchini, Mesquine, Miani. Inter. J. of Control, 1995.
[Blan99] Blanchini. Automatica, 1999.

In-Block Controllability

- The IBC notion was first introduced for finite state machines [CW95].



- The notion was extended to continuous nonlinear systems on closed sets [CW98] and to Automata [HC02].
- These papers did not study conditions for IBC to hold.

[CW95] Caines, Wei. Sys. and Con. Letters, 1995.
[CW98] Caines, Wei. IEEE TAC, 1998.
[HC02] Hubbard, Caines. IEEE TAC, 2002.

IBC of Affine Systems

- $\dot{x} = Ax + Bu + a$ on X $\Leftrightarrow$ $\dot{\tilde{x}} = A\tilde{x} + Bu$ on $\tilde{X}$ satisfying $0 \in \tilde{X}^\circ$

Theorem

Consider the system $\dot{x}(t) = Ax(t) + Bu(t)$ defined on an n-dimensional simplicial polytope X satisfying $0 \in X^\circ$. The system is IBC w.r.t. X if and only if

(i) *(A, B) is controllable.*

(ii) *The so-called invariance conditions of X are solvable (For each $v \in X$, there exists $u \in \mathbb{R}^m$ s.t. $Av + Bu \in C(v)$).*

(iii) *The so-called backward invariance conditions of X are solvable (For each $v \in X$, there exists $u \in \mathbb{R}^m$ s.t. $-Av - Bu \in C(v)$).*

[HC14] MKH, Caines. CDC, 2014.

What about constructing IBC regions?

- Study was initiated for hypersurface affine systems $(m = n - 1)$ [HC15]

[HC15] MKH, Caines. CDC, 2015.

How to Start?

Problem (Construction of IBC Polytopes)

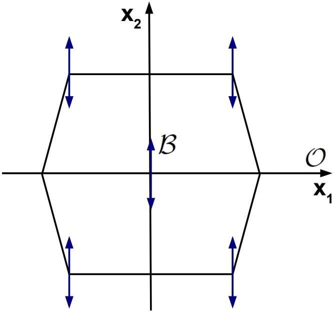
Given a controllable linear system $\dot{x}(t) = Ax(t) + Bu(t)$, construct a polytope X such that $0 \in X^\circ$ and the system is IBC w.r.t. X.

- Straightforward Approach: Construct around the origin a polytope X satisfying both invariance conditions and backward invariance conditions.
- Two difficulties are faced here!
  1. Invariance Cond: For each vertex v of X, there exists $u \in \mathbb{R}^m$ s.t. $h_j \cdot (Av + Bu) \leq 0$. (Given polytopes: Linear Programming (LP) problems; building polytopes satisfying these conditions: Bilinear Matrix Inequalities(BMIs)) NP hard Problem
  2. We still need to verify that the constructed polytope is simplicial!

Geometric Guidelines

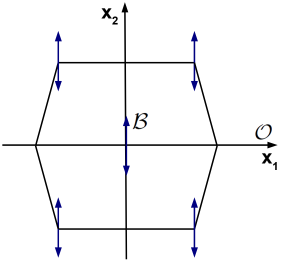
- Let $\mathcal{B}$ be the image of B.
- The set of possible equilibria $\mathcal{O} := \{ x \in \mathbb{R}^n \ : \ Ax + a \in \mathcal{B} \}$

- Result 1: If $v \in \mathcal{O}$ is a vertex of X, then both the inv. and the backward inv. conditions of X are solvable at v.
- Result 2: If $\mathcal{B} \cap C^\circ(v) \neq \emptyset$ at a vertex v, then both the inv. and the backward inv. conditions of X are solvable at v.



Geometric Guidelines

- What about verifying that the constructed polytope is simplicial? No need!



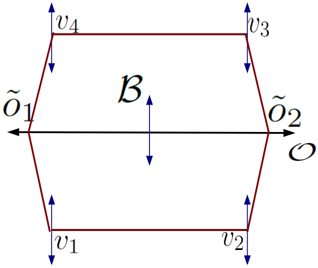
Theorem

Consider a controllable linear system defined on an n-dimensional polytope X satisfying $0 \in X^\circ$. If for each vertex v of X, either $v \in \mathcal{O}$ or $\mathcal{B} \cap C^\circ(v) \neq \emptyset$, then the system is IBC w.r.t. X.

- Punch Line: Construct X such that $v \in \mathcal{O}$ or $\mathcal{B} \cap C^\circ(v) \neq \emptyset$.

Algorithm Idea

- Given: A controllable linear system satisfying $\mathcal{O} + \mathcal{B} = \mathbb{R}^n$



Theorem

Consider a controllable linear system with $\mathcal{O} + \mathcal{B} = \mathbb{R}^n$. Then, the algorithm terminates successfully, and the system is IBC w.r.t. X.

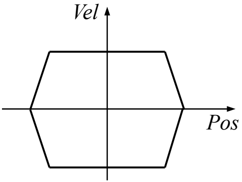
Robot Manipulators

- Consider a robot arm with N links that is modeled by:
  $D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = B(q)\tau$.
- Suppose that $q_i \in [q_{i,min}, q_{i,max}]$, $\dot{q}_i \in [\dot{q}_{i,min}, \dot{q}_{i,max}]$, and $\tau_i \in [\tau_{i,min}, \tau_{i,max}]$.

- Objective: Build a safe speed profile for the robot manipulator.
- For fully-actuated robots, $\tau = B^{-1}(q)(C(q,\dot{q})\dot{q} + g(q) + D(q)u)$ converts the dynamics to the equivalent controllable linear system: $\ddot{q} = u$, a set of decoupled double integrators $\ddot{q}_i = u_i$, where $u_i \in [u_{i,min}, u_{i,max}]$.
- $\mathcal{O} + \mathcal{B} = \mathbb{R}^n \Rightarrow$ Our algorithm can be applied.

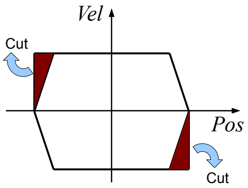
One DOF Robot

- Intuition: Building an IBC region $\Leftrightarrow$ Providing for each position of the robot a corresponding safe speed range, resulting in an overall safe speed profile for the robot.



IBC Region

- Safe speed profiles based on intuition or the controlled invariance property
- Advantages of the proposed approach
- Punch Line: Select the states of the robots' reference trajectories inside the constructed IBC region.



Summary of Results

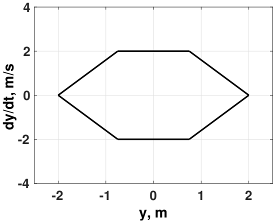
- We reviewed the in-block controllability (IBC) notion, which formalizes Kalman's controllability under safety constraints.
- We introduced the problem of constructing IBC regions.
- We showed the difficulties that are faced if one tries to directly use the existing results for checking IBC to construct IBC regions.
- Following a geometric approach, we proposed a computationally efficient algorithm for constructing IBC regions.
- Used the proposed algorithm for building safe speed profiles for several classes of robotic systems, including robotic manipulators and ground robots.

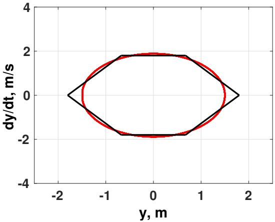
What is next?

- The algorithm can be applied to other classes of robots.
- We use the algorithm for constructing safe speed profiles for unmanned aerial vehicles (UAVs), and then utilize the safe profiles to:
  - achieve static/dynamic obstacle avoidance for UAVs;
  - determine the feasibility of reference trajectories for UAVs.

Dynamic Obstacle Avoidance for UAVs

Safe Position Space





Demo Video of Obstacle Avoidance for UAVs

Demo Video

References

[GC86] P.-O. Gutman, M. Cwikel, Admissible Sets and Feedback Control for Discrete-Time Linear Dynamical systems with Bounded Controls and States, IEEE Transactions on Automatic Control, Vol. 31 (4), pp. 373-376, Apr 1986.

[DH99] C.E.T. Dorea, J.-C. Hennet. (A,B)-invariance Conditions of Polyhedral Domains for Continuous-Time Systems. *European Journal of Control*, vol. 5, pp. 70-81, 1999.

[BMM95] F. Blanchini, F. Mesquine, S. Miani. Constrained Stabilization with Assigned Initial Condition Set. *International Journal of Control*, Vol. 62(3), pp. 601-617, 1995.

[Blan99] F. Blanchini. Set Invariance in Control. *Automatica*, Vol. 35, pp. 1747–1767, 1999.

[CW95] P. E. Caines and Y. J. Wei. The Hierarchical Lattices of a Finite Machine. *Sys. and Con. Letters*, vol. 25, pp. 257-263, 1995.

References

[CW98] P. E. Caines and Y. J. Wei. Hierarchical Hybrid Control Systems: A Lattice Theoretic Formulation. *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 501-508, Apr 1998.

[HC02] P. Hubbard, P. E. Caines. Dynamical Consistency in Hierarchical Supervisory Control. *IEEE Trans. Aut. Con.*, 47(1), pp 37 - 52, 2002.

[HC14] M. K. Helwa, P. E. Caines. In-Block Controllability of Affine Systems on Polytopes. *The 53rd IEEE Conference on Decision and Control*, Los Angeles, Dec 2014, pp. 3936-3942.

[HC14] M. K. Helwa, P. E. Caines. On the Construction of In-Block Controllable Covers of Nonlinear Systems on Polytopes. *The 54th IEEE Conference on Decision and Control*, Osaka, Dec 2015, pp. 276-281.