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Topology-Dependent Stability of a Network of Dynamical Systems with Communication Delays

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***Abstract*— In this paper, we analyze the stability of a network of first-order linear time-invariant systems with constant, identical communication delays. We investigate the influence of both system parameters and network characteristics on stability. In particular, a non-conservative stability bound for the delay is given such that the network is asymptotically stable for any delay smaller than this bound. We show how the network topology changes the stability bound. Exemplarily, we use these results to answer the question if a symmetric or skew-symmetric interconnection is preferable for a given set of subsystems.**

***Index Terms*— Network of dynamical systems, time-delay, network topology, stability.**

I. Introduction

The intersection of control and information technology has become one of the most active fields in the control community. Communication networks offer the possibility to easily connect large sets of dynamical systems. This is particularly interesting when solving complex control tasks using decentralized controllers. Examples can be found in many fields of application, like cooperative control of groups of uninhabited autonomous vehicles (UAV), decentralized control of vast chemical plants, or interconnection of systems and controllers in cars. All these applications are increasingly important, but their rigorous analysis and design come along with a lot of problems. They mainly arise from the big gap between control and information theory.

In the considered area, most of the work on large sets of dynamical systems can be divided in two categories: *cooperative control* and *decentralized control*. The cooperative control of sets of *independent* dynamical systems aims at achieving general tasks like consensus or formation building [1]–[10]. In these works, a local control law for each subsystem is proposed. Then it is shown that a given objective is achieved despite some network deficiencies like packet delays, packet loss, or switching topologies. The design of decentralized controllers for sets of *interconnected* dynamical systems with delays goes back to the 80s. In most cases, it is assumed that the interconnection between the subsystems is delayed and there is no communication between the controllers [11]–[13]. Recent works expand these ideas to decentralized networked controllers [14], [15].

Our work deals with *decentralized networked control* of large sets of dynamical systems. However, we are not considering the design of local controllers. Rather, we study the stability of the obtained *network of dynamical systems* (NDS). How does the stability of this NDS depend on the parameters of the subsystems and of the network? In fact, this question expresses one of the principal difficulties when combining control and information technology: how do they effect each other?

In this paper, we consider an arbitrary number of identical subsystems modeled as first-order LTI systems. The network is represented by a directed graph and exhibits constant and identical communication delays in all channels. Constant communication delays are guaranteed by certain communication protocols. Our main result gives the exact stability bound $\overline{\tau}$ such that this NDS is asymptotically stable for any delay $\tau \in [0, \overline{\tau})$, i.e., the system is not asymptotically stable for $\tau = \overline{\tau}$. We derive an analytical formula that relates $\overline{\tau}$ to the parameters of the subsystems and the eigenvalues of the adjacency matrix of the underlying graph. Exemplarily, we use this result to describe the principal difference between symmetric and skew-symmetric interconnections.

The remainder of this paper is structured as follows: In Section II, we present some basic material on the stability analysis of time-delay systems and on algebraic graph theory. The problem statement is given in Section III. The exact stability bound $\overline{\tau}$ is derived in Section IV. In Section V, we show how this result can be used to investigate different network topologies. The paper is concluded in Section VI.

II. Preliminaries

A. Stability of Time-Delay Systems

A good introduction to time-delay systems (TDS) is given in [16], [17]. More details on the discussed topics can be found there. We consider the following linear time-invariant (LTI) TDS with delay τ

$$\begin{aligned} \dot{x}(t) = &\ A_0\, x(t) + A_1\, x(t-\tau)\,, & t \geq 0\,, \\ x(\theta) = &\ x_0(\theta)\,, & \theta \in [-\tau, 0]\,, \end{aligned} \tag{1}$$

with $x(t) \in \mathbb{R}^n$, $A_0, A_1 \in \mathbb{R}^{n\times n}$, $\tau \geq 0$ and initial condition $x_0(\theta)$, a continuous function mapping $[-\tau, 0]$ to $\mathbb{R}^n$.

The stability of this TDS can be studied efficiently using frequency domain methods. In the same spirit as for delay-free LTI systems, a *characteristic quasipolynomial* $p(s,\, e^{-\tau s})$ of (1) is derived using the Laplace transform:

$$p(s,\, e^{-\tau s}) = \det\left(s\, I - A_0 - A_1 e^{-\tau s}\right). \tag{2}$$

Remember that the stability exponent α_0

$$\alpha_0 := \max\left\{\mathrm{Re}(s) \mid p(s,\, e^{-\tau s}) = 0\right\} \tag{3}$$

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is continuous with respect to τ, see [16], and we have the following result:

Lemma 1 ([16], Thm. 1.5): System (1) *is asymptotically stable if and only if* $\alpha_0 < 0$.

In other words, all roots of (2) must be in the open left half plane $\mathbb{C}_{<0}$. Note that (2) has infinitely many roots. For simplicity, we call a system *stable* if it is asymptotically stable and otherwise *unstable*, as done for example in [16].

We assume that system (1) is stable without delay, i.e., all roots of (2) are in $\mathbb{C}_{<0}$ for $\tau = 0$. We increase τ until the system becomes unstable, i.e. $\alpha_0 = 0$, and define the *stability bound*

$$\overline{\tau} := \min\left\{\tau \geq 0 \,|\, p(j\omega, e^{-j\omega\tau}) = 0 \text{ for some } \omega \in \mathbb{R}\right\} . \tag{4}$$

Obviously, the systems is stable for any $\tau \in [0, \overline{\tau})$ and becomes unstable for $\tau = \overline{\tau}$. Note however that this does not imply that the system is unstable for all $\tau \in [\overline{\tau}, \infty)$. Whenever $\overline{\tau} < \infty$, the system is called *delay-dependent stable*. It is said to be *delay-independent stable*, if (1) is stable for all $\tau \geq 0$.

The pairs (ω_k, τ_k) satisfying the equation

$$p(j\omega_k, e^{-j\omega_k\tau_k}) = 0 \tag{5}$$

have some important properties discussed in [16] and [18]:

- There is only a finite number of possible zero-crossing frequencies ω_k.
- Since (2) is a real quasipolynomial, all its complex roots appear in complex conjugate pairs. Consequently, it suffices to consider only $\omega_k > 0$. Note that the frequency $\omega = 0$ is not a possible zero-crossing frequency because we assume that the system is stable for $\tau = 0$, i.e.,

$$p(j\omega, 1) \neq 0 \quad \forall\omega \quad \Rightarrow \quad p(0, 1) \neq 0 .$$

There are many different ways to compute $\overline{\tau}$. We have studied various of them and the most suitable for the considered problem is the *frequency-sweeping test*, e.g. [16]. The name frequency-sweeping test refers to a stability analysis using the eigenvalues of a frequency-dependent matrix.

Theorem 1 ([16], Thm. 2.1): System (1) *is stable independent of delay if and only if*

(i) A_0 *is stable,*
(ii) $A_0 + A_1$ *is stable, and*
(iii) $\rho\left((j\omega\, I - A_0)^{-1} A_1\right) < 1\,, \quad \forall\, \omega > 0\,,$

where $\rho(\cdot)$ *denotes the* spectral radius *of a matrix.*

Theorem 2 ([16], Thm. 2.2): Suppose that system (1) *is stable at* $\tau = 0$ *and that the matrix* $(j\omega\, I - A_0)$ *is invertible for all* $\omega \in (0, \infty)$. *Furthermore, define*

$$\overline{\tau}_k := \begin{cases} \min\limits_{i=1,\dots,n} \tau_{ki} & \text{if } \lambda_k\left((j\omega_{ki}\, I - A_0)^{-1} A_1\right) = e^{-j\theta_{ki}} \\ & \text{for some } \omega_{ki} \in (0, \infty)\ ,\ \theta_{ki} \in [0, 2\pi] \\ \infty & \text{if } \rho\left((j\omega\, I - A_0)^{-1} A_1\right) < 1\,, \\ & \forall\, \omega \in (0, \infty)\ , \end{cases} \tag{6}$$

where $\theta_{ki} = \omega_{ki}\,\tau_{ki}$. *Then,* $\overline{\tau} = \min_{k=1,\dots,n} \overline{\tau}_k$, *i.e., system* (1) *is stable for all* $\tau \in [0, \overline{\tau})$ *but becomes unstable at* $\tau = \overline{\tau}$.

B. Basic Algebraic Graph Theory

A directed graph $G = (\mathcal{V}, \mathcal{E})$ of order n is defined by a set of vertices $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$, and a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The edges of G are denoted by $e_{ij} = (v_i, v_j) \in \mathcal{E}$, i.e., there is an edge from vertex j to vertex i. We consider directed graphs without loops, i.e., $e_{ii} \notin \mathcal{E}$ for all $i \in \mathcal{I} = \{1, 2, \dots, n\}$. Moreover, we define the adjacency matrix $C = [c_{ij}] \in \mathbb{R}^{n\times n}$. Each adjacency element c_{ij} is associated with an edge $e_{ij},\ i, j \in \mathcal{I}$. We require for all edges $e_{ij} \in \mathcal{E} \iff c_{ij} \neq 0$. More details on algebraic graph theory can be found for example in [19].

Among others, we consider the well-known ring, star, and complete topology, shown in Table I for the case $n = 4$. The eigenvalues of these topologies are given by (cf. [20])

$$\begin{aligned} &[\lambda(C_{\text{ring}})]^n = 1\,, \\ &\lambda_{1,2}(C_{\text{star}}) = \pm\sqrt{n-1}\,, \quad \lambda_{3,\dots,n}(C_{\text{star}}) = 0\,, \\ &\lambda_1(C_{\text{compl}}) = n-1\,, \qquad \lambda_{2,\dots,n}(C_{\text{compl}}) = -1\,, \end{aligned}$$

where $C_{\text{ring}}, C_{\text{star}}, C_{\text{compl}} \in \mathbb{R}^{n\times n}$.

III. Problem Statement

A NDS consisting of n dynamical subsystems can be represented by $\mathcal{S} = (\Sigma, G, \mathcal{T})$. The behavior of the dynamical subsystems is described by $\Sigma = (\Sigma_1, \dots, \Sigma_n)$, e.g., using differential equations. The input and output of each subsystem are u_i and y_i, respectively. The topology of the network is given by the graph G of order n, where the vertex v_i of G corresponds to subsystem Σ_i. The communication delay is represented by $\mathcal{T} = [\tau_{ij}] \in \mathbb{R}^{n\times n}_{\geq 0}$ such that $c_{ij} = 0 \Rightarrow \tau_{ij} = 0$ for all $i, j \in \mathcal{I}$. In the most general case, it may be assumed that τ_{ij} is time-variant or stochastic. A communication channel from Σ_j to Σ_i is modeled by an edge $e_{ij} \in \mathcal{E}$ with amplification c_{ij} and communication delay τ_{ij}. In other words, the input $u_i(t)$ of system Σ_i is a function of the weighted outputs $c_{ij}y_j(t - \tau_{ij})$ of the neighboring systems Σ_j with $e_{ij} \in \mathcal{E}$. The stabilization of such a NDS $\mathcal{S}$ has for example been addressed in [5], yet without delays.

In this paper, we assume that the dynamical representation of all subsystems Σ_i, $i \in \mathcal{I}$, is the same and can be expressed as

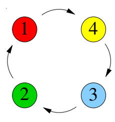
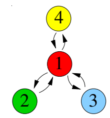
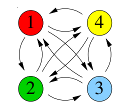
$$\Sigma_i : \begin{cases} \dot{x}_i(t) = a x_i(t) + b u_i(t) \\ y_i(t) = x_i(t) \\ x_i(0) = x_{i,0} \end{cases} \tag{7}$$

with $a,\ b,\ x_i,\ u_i \in \mathbb{R}$. Our main results, Theorems 3 and 4, can be expanded to heterogeneous higher order LTI systems. Yet, the connection between the dynamical behavior of the subsystems and the network topology is much clearer for this first-order model. Moreover, since all subsystems behave in the same way, we can derive stability conditions for arbitrarily large networks depending only on a, b, and n.

Furthermore, we assume that the input $u_i(t)$ of each subsystem depends linearly on the amplified outputs and that

TABLE I

TYPICAL NETWORK STRUCTURES: THE RING, STAR, AND COMPLETE TOPOLOGY FOR 4 VERTICES.

$C_{\text{ring}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ 	$C_{\text{star}} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ 	$C_{\text{compl}} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ 

all delays are constant and identical, $\tau_{ij} = \tau,\ i, j \in \mathcal{I}$, i.e.

$$u_i(t) = \sum_{j=1}^{n} c_{ij} y_j(t-\tau), \quad i \in \mathcal{I}\,. \tag{8}$$

Constant and identical delays are obtained for example using communication protocols with a guaranteed maximal delay. This assumption allows us to determine non-conservative analytical stability bounds $\overline{\tau}$. This is in general not possible for multiple heterogeneous delays because calculating the exact stability bounds of TDS with incommensurate delays is an $\mathcal{NP}$-hard problem [16].

The networked system (7) with inputs (8) can be written in the following compact matrix representation

$$\dot{x}(t) = a\, x(t) + b\, C\, x(t-\tau)\,, \tag{9}$$

where $x \in \mathbb{R}^n$, $C \in \mathbb{R}^{n \times n}$ and $a,\, b,\, \tau \in \mathbb{R},\ \tau \geq 0$. Notice that (9) describes the dynamics of the NDS completely. It contains the information about the dynamical subsystems, a, b, their quantity n, the network topology C, and the communication delay τ. With the frequency-sweeping test, we can now investigate the stability of (9) subject to the parameters a, b, n, and τ as well as the eigenvalues of C.

IV. MAIN RESULT

In this section, we derive the stability bound $\overline{\tau}$ for our NDS (9) based on the frequency-sweeping test introduced in Section II-A. We study the case of delay-independent and delay-dependent stability separately in Sections IV-A and IV-B, respectively. Therefore, we define the delay-independent stability set $S_{DI}(C)$ and the delay-dependent stability set $S_{DD}(C)$ as follows

$$S_{DI}(C) := \{(a,\, b) \mid \text{(9) is delay-independent stable}\}\,.$$

$$S_{DD}(C) := \{(a,\, b) \mid \exists\, \overline{\tau} \in (0, \infty) \text{ such that (9) is stable } \forall \tau \in [0, \overline{\tau}) \text{ but unstable for } \tau = \overline{\tau}\}\,.$$

Note that $S_{DI} \cup S_{DD}$ is the set where (9) is stable for $\tau = 0$.

A. Delay-Independent Stability

The delay-independent stability condition is a direct result of Theorem 1 with $A_0 = a\, I$ and $A_1 = b\, C$.

Theorem 3: NDS (9) *is stable independent of delay if and only if*

(i) $a + b\,\text{Re}\,[\lambda_k\,(C)] < 0\,, \quad \forall\, k \in \mathcal{I}\,$, *and*
(ii) $b^2\, \rho^2\,(C) \leq a^2\,.$

Proof: The first condition of Theorem 1 requires $a < 0$. The second condition transfers to

$$\text{Re}\,[\lambda_k\,(a\,I + b\,C)] < 0 \;\Leftrightarrow\; a + b\,\text{Re}\,[\lambda_k\,(C)] < 0\,, \tag{10}$$

for all $k \in \mathcal{I}$. For the last condition, we have

$$b^2\,\rho^2\,(C) < a^2 + \omega^2\,, \forall\, \omega > 0 \quad \Leftrightarrow \quad b^2\,\rho^2\,(C) \leq a^2\,. \tag{11}$$

Note that $a < 0$ is necessary for both (10) and (11) to be true. ■

B. Delay-Dependent Stability

The delay-dependent stability set is given in the following theorem.

Theorem 4: There exists a $\overline{\tau} \in (0, \infty)$ *such that* (9) *is stable for all* $\tau \in [0, \overline{\tau})$ *and becomes unstable for* $\tau = \overline{\tau}$ *if and only if*

(i) $a + b\,\text{Re}\,[\lambda_k\,(C)] < 0\,, \quad \forall\, k \in \mathcal{I}\,$, *and*
(ii) $b^2\,\rho^2\,(C) > a^2\,.$

Then, $\overline{\tau} = \overline{\tau}(a, b, C)$ *is given by*

$$\overline{\tau} = \min_k \frac{1}{\omega_k} \arccos\left(\frac{\omega_k\, \text{Im}\,(\lambda_k\,(C)) - a\,\text{Re}\,(\lambda_k\,(C))}{b\,|\lambda_k(C)|^2}\right) \tag{12}$$

subject to $b^2\; |\lambda_k(C)|^2 > a^2$, *where*

$$\omega_k = \sqrt{b^2\; |\lambda_k(C)|^2 - a^2}\,. \tag{13}$$

The proof is presented in the appendix.

Remark 1: Remember that Theorem 3 and 4 are only valid if the delays in all channels are identical, i.e., $\tau_{ij} = \tau$ for all $i, j \in \mathcal{I}$. In particular, they are not valid for heterogeneous, i.e. incommensurate, delays $\tau_{ij} \leq \overline{\tau}$.

C. Interpretation

Note that both Theorem 3 and 4 depend only on the eigenvalues of the adjacency matrix C, but not on the matrix itself. Moreover, Equation (12) applies for an arbitrary number n of subsystems. This is remarkable because the eigenvalues of the adjacency matrices of typical network structures can be expressed as functions of n, see Section II-B. Using these results, we can easily compare the stability sets of different topologies with arbitrarily many subsystems. In addition, we can determine how the stability sets of these topologies change with an increasing number of subsystems.

We suppose that an adjacency matrix C is given in order to illustrate the result of Theorem 3. From Conditions (i) and

(ii) in Theorem 3, it results $a < 0$. Furthermore, the lines

$$b = \frac{1}{\rho(C)}a \quad \text{and} \quad b = -\frac{1}{\rho(C)}a \tag{14}$$

bound the stability set $S_{DI}(C)$. The points (a, b) on these boundaries (14) are stable except for the case that the eigenvalue $\lambda^*(C)$ with the highest modulus is real. In this case, all points on the upper boundary $b = -\frac{1}{\rho(C)}a$ are not in S_{DI} if $\rho(C) = \lambda^*(C)$. All points on the lower boundary $b = \frac{1}{\rho(C)}a$ are not in S_{DI} if $\rho(C) = -\lambda^*(C)$. If there are two real eigenvalues $\lambda_1^*(C)$ and $\lambda_2^*(C) = -\lambda_1^*(C)$ with highest modulus, then all points on both boundaries are not in S_{DI}.

Note that the stability set S_{DI} has the same triangular shape for any topology and depends only on the spectral radius $\rho(C)$. The set gets smaller as $\rho(C)$ increases. Two examples of S_{DI} are depicted in the third column of Table II. The delay-independent stability set is dark blue.

Next, we consider the ring, star, and complete topology, presented in Section II-B. By increasing the number of subsystems, the stability set $S_{DI}(C_{\text{ring}})$ does not change, whereas $S_{DI}(C_{\text{star}})$ and $S_{DI}(C_{\text{compl}})$ become smaller. Thereby, the decrease in $S_{DI}(C_{\text{compl}})$ is larger than the decrease in $S_{DI}(C_{\text{star}})$. Similar results are obtained for S_{DD}.

V. Symmetric and Skew-Symmetric Adjacency Matrices

In this section, we exploit our main result to compare the stability of NDS with different interconnections. As an example, we consider symmetric and skew-symmetric adjacency matrices C. After investigating the differences of these matrices theoretically, we explicitly calculate the stability sets and the stability bound $\overline{\tau}$ for two networked systems consisting of three subsystems.

The eigenvalues of both symmetric matrices $C_s = C_s^T$ and skew-symmetric matrices $C_{ss} = -C_{ss}^T$ have specific properties.

Lemma 2 ([21], Cor. 2.5.14): Real symmetric matrices C_s have only real eigenvalues, i.e., $\text{Im}\,(\lambda_k(C_s)) = 0$ *for all $k \in \mathcal{I}$. Real skew-symmetric matrices C_{ss} have only purely imaginary eigenvalues, i.e.,* $\text{Re}\,(\lambda_k(C_{ss})) = 0$ *for all $k \in \mathcal{I}$.*

These properties result in different stability sets S_{DD} and S_{DI} even if all elements of C_s and C_{ss} are the same except for the sign. This is shown in Example 1. First, we show how the calculation of $\overline{\tau}$ can be simplified.

Corollary 1: Suppose a NDS (9) *with symmetric adjacency matrix C_s that fulfills Conditions (i) and (ii) in Theorem 4. If C_s has positive and negative eigenvalues and the maximal and minimal eigenvalue of C_s fulfill $\lambda_{max}(C_s) \neq -\lambda_{min}(C_s)$, then $S_{DD}(C_s) \neq \emptyset$ and the stability bound is*

$$\overline{\tau} = \frac{1}{\sqrt{b^2\lambda_m^2(C_s) - a^2}} \arccos\left(\frac{-a}{b\lambda_m(C_s)}\right), \tag{15}$$

with

$$\lambda_m(C_s) = \left\{\lambda \,:\, |\lambda| = \max_k \left(|\lambda_k(C_s)|\right),\ k \in \mathcal{I}\right\}.$$

Corollary 2: Suppose a NDS (9) *with skew-symmetric adjacency matrix C_{ss} that fulfils Conditions (i) and (ii) in Theorem 4. Then the stability bound $\overline{\tau}$ is*

$$\overline{\tau} = \frac{1}{2\sqrt{b^2\rho^2(C_{ss}) - a^2}} \arccos\left(1 - \frac{2a^2}{b^2\rho^2(C_{ss})}\right). \tag{16}$$

Corollary 1 and 2 result from Theorem 4. The detailed proofs and more characteristics of the stability sets for symmetric and skew-symmetric adjacency matrices are presented in [20]. Note that the stability bound $\overline{\tau}$ is the same for positive and negative b in the skew-symmetric case.

In the following, we use these corollaries and the results of Section IV to analyze exemplarily two NDS with symmetric and skew-symmetric adjacency matrices. Both of them have complete topology.

Example 1: Consider the adjacency matrices

$$C_s = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad C_{ss} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \tag{17}$$

which represent NDS consisting of three subsystems. The network topologies of C_s and C_{ss} are depicted in the second column of Table II. We use a solid line to show that $c_{ij} = 1$ and a dashed line for edges with $c_{ij} = -1$.

The eigenvalues of the matrices (17) are

$$\lambda_{1,2}(C_s) = -1, \quad \lambda_3(C_s) = 2 \tag{18}$$

and

$$\lambda_{1,2}(C_{ss}) = \pm j\sqrt{3}, \quad \lambda_3(C_{ss}) = 0. \tag{19}$$

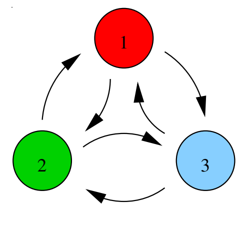
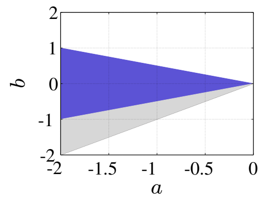
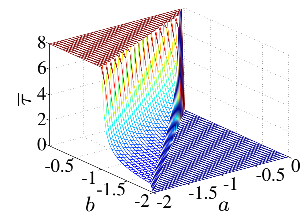
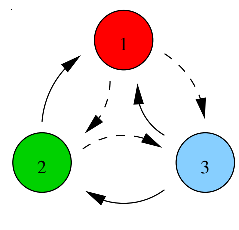
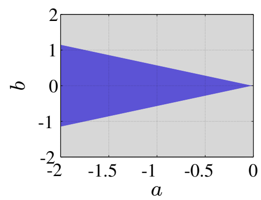
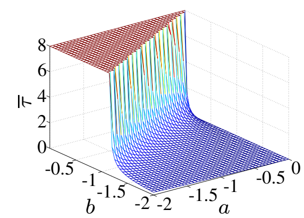
The stability sets S_{DI} (dark blue) and S_{DD} (grey) of both systems are shown in the third column of Table II. Note that there is a small difference in S_{DI} and a big difference in S_{DD}. Both systems are unstable for $a > 0$. In the last column of Table II, we present the stability bounds $\overline{\tau}$ for both systems. In these graphics, all values $\overline{\tau} \geq 8$ are set to $\overline{\tau} = 8$. Remember that in the skew-symmetric case $\overline{\tau}$ is the same for positive and negative b. Therefore, we only present $b \leq 0$. Note that the stability bound $\overline{\tau}$ is higher for the symmetric adjacency matrix as long as a, b remains inside $S_{DD}(C_s)$.

In conclusion, we are able to efficiently analyze the stability of NDS applying the conditions of Section IV. Moreover, we are able to "design" stable NDS: For given parameters a, b, we can determine which topology provides the best stability properties. For Example 1, if we suppose $a = -1$ and $b = -0.75$, then we have $\overline{\tau} \approx 2.06$ for the symmetric case and $\overline{\tau} \approx 1.06$ for the skew-symmetric case. Hence, the symmetric topology is preferable. On the contrary, if we have $a = -1$ and $b = -1.5$, then the skew-symmetric topology guarantees stability up to $\overline{\tau} = 0.165$ and the symmetric topology is unstable even for $\tau = 0$.

Another simple design task is to select an adjacency matrix and input gain b for given a and $\overline{\tau}$. Of course, we could choose C and b such that the networked system lies in S_{DI}. However, we might get a better performance in terms of settling time if we choose a strongly negative b in S_{DD}. For

TABLE II

STABILITY OF NDS (9) – COMPARISON OF A SYMMETRIC VS. A SKEW-SYMMETRIC ADJACENCY MATRIX OF A NETWORK WITH 3 NODES: ADJACENCY MATRIX (FIRST COLUMN), GRAPH TOPOLOGY (SECOND COLUMN), STABILITY SETS S_{DI} (DARK BLUE) AND S_{DD} (GREY) IN THE THIRD COLUMN, AND PLOTS OF THE STABILITY BOUND $\overline{\tau}$ ON THE PARAMETER SET $(a, b) \in [-2, 0] \times [-2, 0]$ (LAST COLUMN).

$C_s = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$			
$C_{ss} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$			

Example 1, we assume $a = -1$ and $\overline{\tau} = 1$ and get $b = -0.76$ for the skew-symmetric adjacency matrix and $b = -0.99$ for the symmetric case. Hence, C_s is the better choice. However, if we have $a = -1$ and $\overline{\tau} = 0.1$, then we get $b = -1.88$ for C_{ss} and again $b = -0.99$ for C_s (remember that this is the bound of $S_{DD}(C_s)$). Obviously, C_{ss} is better suited.

VI. CONCLUSIONS

In this paper, we analyzed the stability of a network of dynamical systems. In particular, we consider an arbitrary number of identical first-order LTI systems that are interconnected via a network with identical communication delays. Our main result gives the exact stability bound $\overline{\tau}$ such that the NDS is asymptotically stable for any delay $\tau \in [0, \overline{\tau})$ but loses this property for $\tau = \overline{\tau}$. The stability bound is derived analytically subject to the parameters of the subsystems and the eigenvalues of the adjacency matrix C of the network. For typical network topologies, these eigenvalues have been determined as a function of n. Finally, we studied the principal difference of symmetric and skew-symmetric adjacency matrices in terms of stability.

Our results apply for a general network of first-order LTI systems and determine its stability bound. An expansion to NDS that consist of non-identical higher-order LTI systems is easily possible at the expense of the compact formulation of the result.

APPENDIX
PROOF OF THEOREM 4

First, we proof the existence of $\overline{\tau}$. Condition (i) is obvious because (9) must be stable for $\tau = 0$. Condition (ii) guarantees that $\overline{\tau} < \infty$ (cf. Theorems 2 and 3).

Next, we show how (12) is derived from Theorem 2. Note that $(j\omega\, I - A_0)$ is invertible for all $\omega \in (0, \infty)$ because we have $A_0 = a\, I\,,\ a \in \mathbb{R}$. We first calculate the zero-crossing frequencies ω_{ki} using the equation

$$\left| \lambda_k \left((j\omega_{ki}\, I - A_0)^{-1} A_1 \right) \right| = 1\,. \qquad (20)$$

Remember that there is only a finite number of zero-crossing frequencies ω_{ki} (Part II-A). We rewrite (20) using (9) and obtain

$$\omega_{ki} = \omega_k = \sqrt{b^2\ |\lambda_k(C)|^2 - a^2}\,, \quad k \in \mathcal{I}. \qquad (21)$$

The radicand is positive for at least one $k \in \mathcal{I}$ because of Condition (ii). Hence, we obtain at least one real $\omega_k > 0$.

We calculate τ_k for each real ω_k using (6) and obtain

$$\omega_k \tau_k = \arg\left[\lambda_k \left((j\omega_k - a)^{-1}\, b\, C \right)\right] \qquad (22)$$

$$= \underbrace{\arg\left[-ab - j\omega_k b\right] + \arg\left[\lambda_k\left(C\right)\right]}_{\in[0,\ 2\pi]}, \qquad (23)$$

where $\arg(z)$ denotes the argument of a complex number z.

Next, we derive (12) using Condition (i), some trigonometric identities, and the fact that all complex eigenvalues of C appear in complex conjugate pairs. For simplicity of notation, $\lambda_k(C)$ is replaced by λ, ω_k by ω, and τ_k by τ. The calculation of the arguments in (23) needs a case differentiation. For $\arg\left[-ab - j\omega b\right]$, the cases shown in Table III must be considered. The argument $\arg\left[-ab - j\omega_k b\right]$ is zero for $b = 0$. However, this case is not of interest because $b = 0$ means that there is no interconnection between the subsystems. Results for $a = 0$ can be determined by calculating the limit $a \rightarrow 0$ in the cases $1 - 4$ (Table III).

For $\arg\left[\lambda\right]$, we have to differentiate the cases presented in Table IV. Results for $\mathrm{Re}(\lambda) = 0$ can be determined by calculating the limit $\mathrm{Re}(\lambda) \rightarrow 0$ in the cases A – D (Table IV).

TABLE III
CASE DIFFERENTIATION FOR THE PARAMETERS a, b IN THE CALCULATION OF (23)

Case	Parameter	$\arg[-ab - j\omega b]$
1	$a < 0\,,\, b < 0$	$\arctan\left(\frac{\omega}{a}\right) + \pi$
2	$a < 0\,,\, b > 0$	$\arctan\left(\frac{\omega}{a}\right)$
3	$a > 0\,,\, b < 0$	$\arctan\left(\frac{\omega}{a}\right)$
4	$a > 0\,,\, b > 0$	$\arctan\left(\frac{\omega}{a}\right) + \pi$

TABLE IV
CASE DIFFERENTIATION FOR THE EIGENVALUES $\lambda_k(C)$ IN THE CALCULATION OF (23)

Case	Parameter	$\arg[\lambda]$
A	$\mathrm{Re}(\lambda) < 0\,,\, \mathrm{Im}(\lambda) \leq 0$	$\arctan\left(\frac{\mathrm{Im}(\lambda)}{\mathrm{Re}(\lambda)}\right) + \pi$
B	$\mathrm{Re}(\lambda) < 0\,,\, \mathrm{Im}(\lambda) > 0$	$\arctan\left(\frac{\mathrm{Im}(\lambda)}{\mathrm{Re}(\lambda)}\right) + \pi$
C	$\mathrm{Re}(\lambda) > 0\,,\, \mathrm{Im}(\lambda) \leq 0$	$\arctan\left(\frac{\mathrm{Im}(\lambda)}{\mathrm{Re}(\lambda)}\right)$
D	$\mathrm{Re}(\lambda) > 0\,,\, \mathrm{Im}(\lambda) > 0$	$\arctan\left(\frac{\mathrm{Im}(\lambda)}{\mathrm{Re}(\lambda)}\right)$

Exemplarily, we consider the combination case 3-D. Equation (21) gives $\omega^2 = b^2\lambda^2 - a^2$ and we obtain

$$\begin{aligned} \tau\,\omega &= \arctan\underbrace{\left(\frac{\omega}{a}\right)}_{>0} + \arctan\underbrace{\left(\frac{\mathrm{Im}(\lambda)}{\mathrm{Re}(\lambda)}\right)}_{>0} \\ &= \arccos\left(\frac{-a}{b\,|\lambda|}\right) + \arccos\left(\frac{\mathrm{Re}\,(\lambda)}{|\lambda|}\right) \\ &= \arccos\left(\frac{\omega\,\mathrm{Im}\,(\lambda) - a\,\mathrm{Re}\,(\lambda)}{b\,|\lambda|^2}\right). \end{aligned} \tag{24}$$

For other combinations, the procedure is the same. Because of Condition (i) and the fact that the eigenvalues of C have complex conjugate pairs, some combinations do not have to be considered. For example, the inequality $\mathrm{Re}(\lambda) > 0$ must hold in case 3 (Condition (i)). All possible combinations result in Equation (24). The time-delay $\overline{\tau}$ is the minimum of all calculated τ_k.

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