Preventing Inactive CBF Safety Filters Caused by Incorrect Relative Degree Assumptions

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***Abstract*—Control barrier function (CBF) safety filters emerged as a popular framework to certify and modify potentially unsafe control inputs, for example, provided by a reinforcement learning agent or a nonexpert user. Typical CBF safety filter designs assume that the system has a uniform relative degree. This assumption is restrictive and is frequently overlooked in practice. When violated, the assumption can cause the safety filter to become inactive, allowing large and possibly unsafe control inputs to be applied to the system. In discrete-time implementations, the inactivity issue is often manifested as chattering close to the safety boundary and/or constraint violations. In this work, we provide an in-depth discussion on the safety filter inactivity issue, propose a mitigation strategy based on multiple CBFs, and derive an upper bound on the sampling time for safety under sampled-data control. The effectiveness of our proposed method is validated through both simulation and quadrotor experiments.**

***Index Terms*—Control barrier function (CBF), nonlinear control, safety, sampled-data control.**

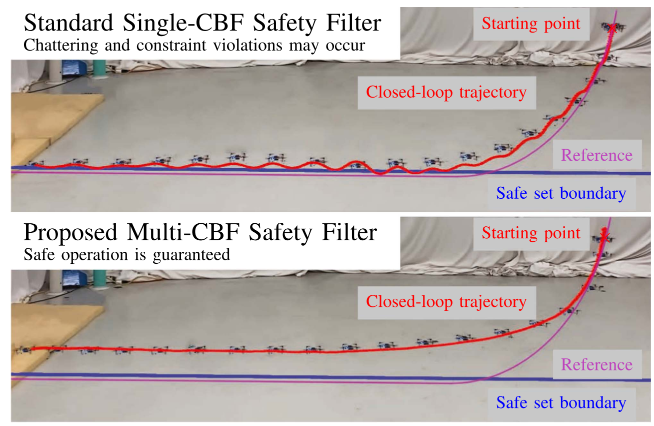

Fig. 1. Real-world quadrotor demonstration with nonlinear dynamics (safe set boundary in blue and trajectory in red). Single CBF (top) results in an inactive safety filter, causing chattering and safe set violations. Our multi-CBF approach (bottom) successfully keeps the safety filter active and the system safe. See Fig. 4 for details.

I. Introduction

Safety is paramount in any real-world application of control systems, such as robotics [1]. A safe set encodes states the system should stay within (i.e., a quadrotor avoiding collision with the floor; see Fig. 1). While most control systems are designed with a focus on safety, many recent learning-based control methods (e.g., reinforcement learning) typically do not provide safety guarantees [1]. Similarly, teleoperating a control system may lead to unsafe scenarios, especially when the user is not an expert. In such cases, the system's safety can be ensured by employing a safety filter, which certifies and, if necessary, modifies an unsafe control input before it is applied to the control system.

A popular line of work on safety filters uses control barrier functions (CBFs) [2]. Based on Nagumo's theorem [3], CBFs [4], [5] provide a scalar condition to check control invariance, which can be efficiently incorporated into a safety filter. CBF safety filters gained popularity due to 1) their applicability to control affine systems, resulting in an efficiently solvable quadratic program (QP), and 2) the removal of overly conservative restrictions that kept the system inside shrinking safe sets [5]. Furthermore, CBFs have already been successfully applied to many real-world safety-critical systems (e.g., safe quadrotor flight [6] or robot swarms [7]).

The relative degree of a system plays an important role in CBF safety filters. The relative degree of a system indicates which time derivative of the system's output can be directly impacted by the control input [8]. This local property may not be constant over a nonlinear system's domain. An active first-order CBF safety filter, where the CBF condition further constrains the set of feasible control inputs, requires the system's relative degree to be one in the entire domain [9]. If this requirement is not met, then the control input cannot impact the first time derivative of the CBF at this state, and the CBF safety filter reduces to projecting control inputs onto the static feasible input set. This relative degree assumption has been relaxed by introducing higher order CBFs, applicable to systems with a relative degree greater than one [10], [11]. These methods, however, still assume a well-defined, uniform relative degree over the entire domain. The assumption of uniform relative degree for first- and high-order CBFs is not always verified in practice and poses limitations. While Jankovic [12] relaxed this assumption by relying on drift dynamics, Alyaseen et al. [13] showed that control policy boundedness and continuity can degrade where this assumption fails, and proved it is necessarily violated for compact safe sets defined by smooth CBF superlevel sets. Tan et al. [14] proposed requiring uniform relative degree only on the safe set boundary, which is still limiting as it does not apply to simple cases like linear dynamics with quadratic CBFs [13]. Recently, Brunke et al. [15] proposed practical approaches for CBFs with nonuniform relative degree, although without

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theoretical analysis. Building on the insights from these works, we formally define conditions to prevent inactive CBF safety filters and develop our mitigation strategy. Another key challenge in applying CBF safety filters is their discrete-time implementation. Although designed for continuous time, real-world systems require applying control inputs at discrete intervals due to delays and computation time, forming a sampled-data system [16]. This can worsen issues like violating relative degree assumptions, as feasible large inputs in magnitude may persist during each sampling interval. Two main approaches address this. First, discrete-time CBFs [17], which ensure safety directly in discrete time but often involve nonconvex optimization, increasing computational load. Second, continuous-time CBFs with sampled-data adjustments, which either add conditions for fixed sampling [18], [19] or adapt sampling online via self-triggering [20], [21]. These methods rely on bounding the deviation between the true trajectory and the last sampled state. In this work, we use a fixed sampling time with a continuous-time formulation. This allows us to exploit the efficient QP formulation and provide safety guarantees for all time rather than just at discrete time steps. Furthermore, we select the minimal feasible sampling time to track the uncertified control input as closely as possible.

In this work, we consider the interplay of these challenges: how the violation of uniform relative degree assumptions leads to inactive safety filters and how their sampled-data implementation causes safe set violations and chattering (i.e., high-frequency changes in the control input due to the safety filter) [22]. To prevent safe set violations, we present a theoretically sound mitigation method based on multiple CBFs in the safety filter. While multiple CBFs have been considered in the literature, they have generally been used for different purposes, such as interagent collision avoidance in multiagent systems [7] or avoiding multiple obstacles [23], [24]. Our work uniquely applies them to address inactive CBF safety filters caused by nonuniform relative degree in sampled-data systems. We summarize our contributions as follows.

1) We analyze CBF safety filter inactivity and examine how feasible input direction and magnitude at such states can cause safety violations or chattering in discrete time.
2) We propose a multi-CBF safety filter along with a synthesis procedure to avoid inactivity and thereby mitigate the resulting chattering and constraint violation issues.
3) We derive a theoretical upper bound on the sampling time for guaranteeing safety under sampled-data implementation.
4) We validate our approach through simulation and real-world quadrotor experiments.

The rest of this article is organized as follows. Section II introduces notation, Section III formulates the problem, and Section IV provides background. Our method is presented in Section V, with evaluations in Section VI. Finally, Section VII concludes this article.

II. Mathematical Preliminaries

In this section, we introduce the notation and general definitions used in the remainder of this work. Let $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_0$ denote the sets of integers, positive integers, and nonnegative integers, respectively. A set of consecutive integers is $\mathbb{Z}_{a,b}$, where $a, b \in \mathbb{Z}$ and $a \leq b$. The Euclidean norm is $\|\cdot\|$. We denote the diagonal matrix $D \in \mathbb{R}^{n\times n}$ with diagonal entries given by a vector $d \in \mathbb{R}^n$ as $D = \mathrm{diag}(d) = \mathrm{diag}(d_1, \ldots, d_n)$ and the diagonal entries as $d = \mathrm{diag}(D)$. The set $\mathbb{B}(x_0, r)$ denotes the closed 2-norm ball of radius r centered at $x_0 \in \mathbb{R}^n$. The Lie derivative of a differentiable function h along a vector field f is denoted by $\mathcal{L}_f h(x) = \frac{\partial h(x)}{\partial x} f(x)$.

In this work, we consider a control-affine system

$$\dot{x}(t) = f(x(t)) + g(x(t))\, u(t) \tag{1}$$

where $t \in \mathbb{R}$ is the time variable, $x \in \mathbb{D} \subset \mathbb{R}^n$ is the state of the system with $\mathbb{D}$ being the set of admissible states, $u \in \mathbb{U} \subset \mathbb{R}^m$ is the input of the system with $\mathbb{U}$ being the set of admissible inputs, and $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $g : \mathbb{R}^n \mapsto \mathbb{R}^{n\times m}$ are locally Lipschitz continuous functions. Hereafter, we omit the dependence on t unless unclear from context.

Definition 1 (Positively control invariant set): Let $\mathfrak{U}$ be the set of all bounded control signals $\nu : \mathbb{R}_{\geq 0} \to \mathbb{U}$. A set $\mathbb{C} \subseteq \mathbb{D}$ is a positively control invariant set for the control system in (1) if $\forall\, x_0 \in \mathbb{C}$, $\exists\, \nu \in \mathfrak{U}$, and $\forall\, t \in \mathbb{T}^+_{x_0}$, $\phi(t, x_0, \nu) \in \mathbb{C}$, where $\phi(t, x_0, \nu)$ is the system's phase flow starting at x_0 under the control signal ν, and $\mathbb{T}^+_{x_0}$ is the maximum time interval.

In this work, we also refer to a positively control invariant set as a safe set and to positive control invariance as safety.

Definition 2 (Extended class-$\mathcal{K}$ function [5]): A function $\gamma : \mathbb{R} \to \mathbb{R}$ is said to be of class-$\mathcal{K}_e$ if it is continuous, $\gamma(0) = 0$, and strictly increasing.

Definition 3 (Least relative degree and relative degree): Consider a set $\mathbb{D} \subset \mathbb{R}^n$ and a system consisting of the dynamics equation in (1) and the output equation $y = h(x)$, where h is ρth-order differentiable. The system is said to have a *least relative degree* of $\rho \in \mathbb{Z}_{1,n}$ over $\mathbb{D}$ if $\mathcal{L}_g \mathcal{L}^i_f h(x) = 0$ for $i \in \mathbb{Z}_{0,\rho-2}$ [14, Def. 4]. Furthermore, the system is said to have a *relative degree* of $\rho \in \mathbb{Z}_{1,n}$ over $\mathbb{D}$ if $\mathcal{L}_g \mathcal{L}^{\rho-1}_f h(x) \neq 0$ for all $x \in \mathbb{D}$ is additionally satisfied [8, Def. 13.2].

III. Problem Formulation

We consider the control-affine system in (1). We assume that the domain $\mathbb{D}$ is an open set, and f and g have Lipschitz constants L_f and L_g, respectively. We are given a bounded desired safe set $\mathbb{X} \subseteq \mathbb{D}$, in which we want to keep the system for all future time $t \geq 0$. We aim to find a compact inner approximation of the desired safe set, $\mathbb{C}_{\mathbb{Z}_{1,K}} \subseteq \mathbb{X}$, where $K \in \mathbb{N}$, $\mathbb{C}_{\mathbb{Z}_{1,K}} = \{x \in \mathbb{R}^n \,|\, h_i(x) \geq 0 \forall i \in \mathbb{Z}_{1,K}\}$ that is control invariant under nonempty compact polytopic input constraints $\mathbb{U} = \{u \in \mathbb{R}^m | A_u u \leq b_u\}$ with continuously differentiable functions h_i for all $i \in \mathbb{Z}_{1,K}$. The boundary of the safe set is $\partial\mathbb{C}_{\mathbb{Z}_{1,K}} = \{x \in \mathbb{R}^n \,|\, \min_{i\in\mathbb{Z}_{1,K}} h_i(x) = 0\}$.

Our goal is to augment a given, potentially unsafe state-feedback controller $\pi : \mathbb{R}^n \to \mathbb{R}^m$ with a safety filter $\pi_{\text{cert}} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ such that the system is safe (i.e., the system's state x stays inside a safe set $\mathbb{C}_{\mathbb{Z}_{1,K}}$ if it starts inside of $\mathbb{C}_{\mathbb{Z}_{1,K}}$). We assume that the controller π is locally Lipschitz continuous and has a Lipschitz constant L_π.

In this work, we aim to guarantee safety for a sampled-data system, where the control input can only be updated discretely at fixed time intervals $t_k = k\Delta t$ with $k \in \mathbb{N}_0$ and $\Delta t > 0$. In a zero-order-hold (ZOH) fashion, the control input is

$$u(t_k + \tau) = u_{\text{cert},k} = \pi_{\text{cert}}\,(x(t_k), u_k) \quad \forall \tau \in [0, \Delta t) \tag{2}$$

with the uncertified control input $u_k = \pi(x(t_k))$, and $u_{\text{cert},k}$ is the certified control input. This setting is motivated by real-world systems, where control inputs cannot be applied arbitrarily fast to the system. A safety filter based on a single CBF ($K = 1$) can contain states on the boundary of $\mathbb{C}_{\{1\}}$ for which the safety filter will become inactive. This is a known issue that reduces the CBF safety filter to projecting the uncertified control input $\pi(x)$ onto the set $\mathbb{U}$, and the CBF has no effect. This inactivity can often lead to safety constraint violations. Therefore, the goal in our work is to propose a mitigation strategy that resolves the inactivity issue.

IV. Background

This section introduces the relevant definitions and background on CBF-based safety filters to facilitate our discussion.

Definition 4 (CBF [14]): Consider the control system in (1) and a first-order differentiable function $h : \mathbb{R}^n \to \mathbb{R}$. The function h is called a CBF, if there exists a class-$\mathcal{K}_e$ function γ, an open set $\mathbb{D}$ with $\mathbb{C} \subset \mathbb{D} \subset \mathbb{R}^n$, where $\mathbb{C}$ is the zero superlevel set of h, the system is of least relative degree $\rho = 1$, and for all $x \in \mathbb{D}$

$$\max_{u \in \mathbb{U}} \left[\mathcal{L}_f h(x) + \mathcal{L}_g h(x) u\right] \geq -\gamma(h(x)) \,. \tag{3}$$

Using a CBF, we can define an input set

$$\mathbb{U}_{h,\gamma}(x) = \{u \in \mathbb{U} \mid \mathcal{L}_f h(x) + \mathcal{L}_g h(x) u \geq -\gamma(h(x))\} \tag{4}$$

that renders the system safe [5].

Theorem 1 (Control invariance of $\mathbb{C}$ [14]): Consider a CBF h as defined in Definition 4. Then, any locally Lipschitz continuous controller $\pi : \mathbb{R}^n \to \mathbb{R}^m$ such that $\pi(x) \in \mathbb{U}_{h,\gamma}(x)$ for all $x \in \mathbb{C}$ will render the set $\mathbb{C}$ positively control invariant for the system (1).

For a policy $\pi(x)$ that is not initially designed to be safe, one can formulate a QP to modify the control input such that the system is guaranteed to be safe [25]

$$\pi_{\text{cert}}(x, u_k) = \underset{u \in \mathbb{U}_{h,\gamma}(x)}{\operatorname{argmin}} \quad \frac{1}{2}\|u - u_k\|^2. \tag{5}$$

Intuitively, a safety filter finds an input in $\mathbb{U}_{h,\gamma}(x)$ that best matches u_k, based on a chosen distance measure [e.g., the Euclidean norm in (5)].

We model the discretization error introduced by the ZOH as a disturbance. The closed-loop sampled data system is

$$\dot{x}(t) = f(x(t)) + g(x(t))\pi(x(t_k)) \tag{6}$$

where $t_k = k\Delta t$ with $k \in \mathbb{N}_0$. Based on the results in [26], a control policy $\pi(x)$ yields an input-to-state safe closed-loop system on $\mathbb{C}_d = \{x \in \mathbb{R}^n \mid h_d(x) = h(x) - d \geq 0\} \neq \emptyset$ for the sampled-data implementation in (6) if the following holds:

$$\begin{aligned} &\mathcal{L}_f h_d(x(t)) + \mathcal{L}_g h_d(x(t))\pi(x(t_k)) \geq \\ &\quad - \gamma(h_d(x(t))) - L_\pi M e(t, x(t_k), \pi(x(t_k))) \end{aligned} \tag{7}$$

where $t = t_k + \Delta t$, $M > 0$ is such that $\|\mathcal{L}_g h(x)\| = \|\mathcal{L}_g h_d(x)\| \leq M$, and $e(t, x(t_k), \pi(x(t_k))) = \|x(t) - x(t_k)\|$.

Theorem 2 (Invariance of sampled-data systems [21]): Let h be a continuously differentiable function, and let the closed-loop sampled data system in (6) be Lipschitz continuous with a state-feedback control policy π. Furthermore, let $\pi(x)$ yield an input-to-state safe closed-loop sampled data system on $\mathbb{C}_d$. For each $d \in (0, h_{\max})$, where $h_{\max} = \max_{x \in \mathbb{C}} h(x)$, there exists an upper bound $e_{\max} = -\frac{\gamma(-d)}{L_\pi M} > 0$ on the sample-and-hold error such that if $\|e\|_\infty \leq e_{\max}$, the set $\mathbb{C}$ is positively control invariant under the control policy $\pi(x)$.

In [21], Lipschitz continuity of the certified control policy in the sampled-data implementation is assumed, which can be restrictive in practice. This assumption can be relaxed by tightening the CBF condition in (3) based on the sampling time and the system dynamics to still achieve control invariance [19]. We further note that while we focus on the sampled-data effect, the notion of input-to-state safety can be extended to other types of practical disturbances (e.g., model mismatch and measurement noise) [26].

V. Methodology

In this section, we begin by investigating the cause and consequence of inactive CBF safety filters. Based on the insights, we then propose a method to mitigate inactive safety filter issues and detail how safety can be achieved in practice.

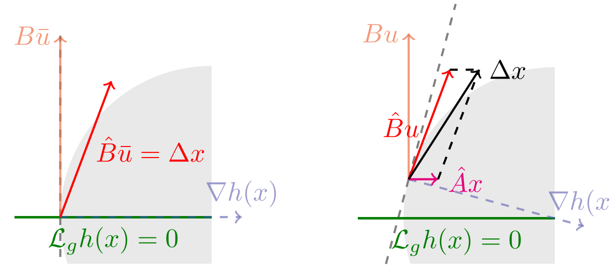

Fig. 2. Effect of $\|\mathcal{L}_g h(x)\|$ on safety filter activity for a double integrator system on an ellipsoidal safe set (gray). We use $\Delta t = 0.75\,\text{s}$ and uncertified policy $\pi(x) = \bar{u} = \max_{\mathbb{U}} \|u\|$. When $\mathcal{L}_g h(x) = 0$ (safety filter inactive, left), $\pi(x) = \bar{u}$ is allowed. When $\|\mathcal{L}_g h(x)\| \neq 0$ (right), $\bar{u}$ is not allowed, although the certified input u remains large. Both cases can cause safe set violations in sampled-data implementation.

A. Inactive CBF Safety Filters

1) Conditions for Inactivity: We refer to a CBF safety filter being inactive at state x when the constraint (3) has no effect on the resulting certified control policy $\pi_{\text{cert}}(x)$, i.e., the constraint is redundant with $\mathbb{U}_{h,\gamma}(x) = \mathbb{U}$. At such states, the safety filter reduces to a projection of the policy $\pi(x)$ onto the control input set $\mathbb{U}$. Clearly, the safety filter is inactive at x when $\mathcal{L}_g h(x) = 0$. Then, one of the following is satisfied: 1) $g(x) = 0$ (the system is autonomous at x), 2) $\frac{\partial h(x)}{\partial x} = 0$, or 3) $\frac{\partial h(x)}{\partial x} \perp g(x)$ (the gradient of h is orthogonal to every column in $g(x)$). The Lie derivative $\mathcal{L}_g h(x)$ being zero is related to the system's relative degree at x. For convenience, denote the states where the relative degree $\rho = 1$ as $\mathbb{D}_{\rho=1} = \{x \in \mathbb{R}^n \mid \mathcal{L}_g h(x) \neq 0\}$. We also define the set of states where the relative degree is not equal to one as $\mathbb{D}_{\rho \neq 1} = \mathbb{D} \setminus \mathbb{D}_{\rho=q}$.

Besides $\mathcal{L}_g h(x) = 0$, the safety filter may also be effectively inactive when $\mathcal{L}_g h(x)$ has small magnitude. To show this, we rewrite the CBF condition as $\alpha^\intercal(x) u \leq \beta(x)$, where $\alpha^\intercal(x) = -\frac{\mathcal{L}_g h(x)}{\|\mathcal{L}_g h(x)\|}$ with $\|\alpha\| = 1$ and $\beta(x) = \frac{\gamma(h(x)) + \mathcal{L}_f h(x)}{\|\mathcal{L}_g h(x)\|}$. This reveals that every CBF yields a state-dependent affine control input constraint in standard affine form. The support function $\sigma_{\mathbb{U}}(\alpha) = \max_{u \in \mathbb{U}} \alpha^\intercal u$ is a linear program that measures how far the set $\mathbb{U}$ extends in the direction of α. Consequently, the safety filter is inactive if $\beta(x) > \sigma_{\mathbb{U}}(\alpha(x))$ or, equivalently, if $\|\mathcal{L}_g h(x)\| < \frac{\gamma(h(x)) + \mathcal{L}_f h(x)}{\sigma_{\mathbb{U}}(\alpha(x))} =: \epsilon(x)$.

2) Impacts of Inactive CBF Safety Filters: Inactive safety filters have several practical implications. First, as highlighted above, for states $\bar{x} \in \mathbb{C}$ such that $\mathcal{L}_g h(\bar{x}) = 0$, the gradient of h is orthogonal to every column $g(\bar{x})$. The input matrix $g(\bar{x})$ projects the control input u onto a subspace of $\mathbb{D}$ spanned by the columns of $g(\bar{x})$ that is tangential to the level set of $h(x)$ at $\bar{x}$. For any convex zero superlevel set $\mathbb{C}_d$, where $d = h(\bar{x})$, the input's contribution to the system dynamics $g(\bar{x})u$ at $\bar{x}$ points outside of $\mathbb{C}_d$ for any $u \neq 0$. In particular, for any $\bar{x} \in \partial\mathbb{C}$ and $u \neq 0$, we have that $g(x)u$ will point outside of the safe set $\mathbb{C}$; see Fig. 2. The drift term $f(x)$ may counteract or support this direction, depending on the system dynamics. We note that if any element of $\mathcal{L}_g h(x)$ is 0, then the same issue would also arise for the corresponding input channel.

Second, if the safety filter is inactive at $\bar{x}$, any control policy $\pi(\bar{x}) = u_{\text{cert}} \in \mathbb{U}$ is feasible; see Fig. 2. Again, if any element of $\mathcal{L}_g h(x)$ is 0, then the associated element of the input vector is also not constrained by the CBF. An example of a feasible but aggressive control policy is a policy that tries to maximize the magnitude of $g(x)u$ with $\pi_+(x) = \arg\max_{u \in \mathbb{U}} \|g(x)u\| \in \mathbb{U}$. The control policy $\pi_+(\bar{x})$ is certified by the optimization problem in (5) since it is feasible. If the control input set $\mathbb{U}$ is large, i.e., the control input constraints are loose, then the control input $u_{\text{cert}} = \pi_+(\bar{x})$ is also large in magnitude. Since the set $\mathbb{U}$

is time- and state-invariant, it already has to be chosen sufficiently large such that at least the CBF condition in (3) is feasible for all $x \in \mathbb{C}$.

Furthermore, while the safety filter is formulated for a continuous-time system, its implementation in real-world scenarios is in discrete time. This yields a sampled-data setting since we can only apply control inputs at time intervals with duration $\Delta t > 0$. The state $x(t_k + \tau)$ is computed under the application of $u_{\text{cert},k}$ at time t_k on the interval $\tau \in [0, \Delta t)$. At time t_k, the control input $u_{\text{cert},k} \in \mathbb{U}_{h,\gamma}(x(t_k))$ is certified for state $x(t_k)$ using (5).

In the following, we discuss the consequences of the above. The sampled-data implementation affects the safety filter at all states x for all policies π. This in itself can already lead to unsafe states, $x \notin \mathbb{C}$. While $u_{\text{cert},k}$ is certified to be a safe control input at time t_k, it may not be a safe control input for $x(t_k + \tau)$ for $\tau \in (0, \Delta t)$. Now, consider a state $\bar{x}$ where the safety filter is inactive and $\bar{x} \in \partial\mathbb{C}$. In this case, any nonzero control input u contributes to $\dot{x}$ at $\bar{x}$ pointing outside of the safe set $\mathbb{C}$. In combination with a sampled-data implementation, this can lead to taking a step in an unsafe direction for a time step of $\Delta t > 0$. This can be seen from the following example. Consider a double integrator system of the form $\dot{x} = Ax + Bu$. The CBF candidate is $h(x) = 1 - x^\intercal P x$, where $P = \text{diag}(p_1, p_2) \succ 0$. The resulting safe set is $\mathbb{C} = \{x \in \mathbb{R}^2 \mid h(x) \geq 0\}$, which is an ellipsoid. Exact-discretization with a ZOH control input $u(t_k) = u_k \in \mathbb{R}$ yields $x(t_k + \Delta t) = x_{k+1} = \hat{A}x_k + \hat{B}u_k$, where $\hat{A} = \exp(A\Delta t)$, $\hat{B} = (\int_0^{\Delta t} \exp(A\tau)d\tau)B$, and $\Delta t > 0$. For $x_k = [-1/\sqrt{p_1} \quad 0]^\intercal$, $h(x_k) = 0$, and $\mathcal{L}_g h(x_k) = 0$, the change in h is $h(x_{k+1}) - h(x_k) = h(x_{k+1}) = \eta u_k^2 + \zeta u_k$ with $\eta = -(\Delta t)^2(p_2 + (\Delta t)^2 p_1/4)$ and $\zeta = (\Delta t)^2\sqrt{p_1}$, which is quadratic in u_k. For any $u_k < 0$ or $u_k > \frac{4\sqrt{p_1}}{4p_2 + p_1(\Delta t)^2}$, under the assumption of $u_k \in \mathbb{U}$, this yields $h(x_{k+1}) < 0$, such that the system leaves the safe set; see Fig. 2. Finally, we examine the scenario where, in addition to the above, the desired control policy also minimizes $h(x_{k+1})$. For the double integrator example, either the largest or the smallest feasible control input $u_{\text{cert},k} \in \mathbb{U}$ minimizes the value of h at time step $(k+1)$. This highlights how inactive safety filters can lead to safe set violations. Once the system reaches a state at which $\|\mathcal{L}_g h(x)\|$ is sufficiently large, the safety filter becomes active again and guides the system back into the safe set $\mathbb{C}$. If this behavior repeats multiple times, the system is exhibiting chattering, which induces high-frequency control inputs that are typically undesirable in real-world systems. We briefly note that these effects also apply to other states where the safety filter is inactive and not just $\mathcal{L}_g h(x) = 0$ (e.g., for states where $\|\mathcal{L}_g h(x)\| \leq \epsilon(x)$). Furthermore, states with active safety filters that are in a neighborhood of the boundary may also yield safety violations.

3) Prevalence of Inactive CBF Safety Filters: After discussing the causes and the undesirable effects of inactive safety filters, we next highlight for which systems and CBF candidates the problem of inactive safety filters may be encountered. As it turns out, inactive safety filters are common for many physical systems with compact safe sets $\mathbb{C}$. Our next example shows that a commonly chosen CBF in the literature does not have relative degree $\rho = 1$ but rather least relative degree $\rho = 1$ on $\mathbb{D}$, which results in safety filter inactivity. Consider linear system dynamics of the form $\dot{x} = Ax + Bu$. The CBF candidate is $h(x) = 1 - x^\intercal P x$, where P is positive definite. For the first-order Lie derivative with respect to g, we have $\mathcal{L}_g h(x) = -2x^\intercal PB$, which is state-dependent. According to Definition 3, the set of states where the relative degree is one and not equal to one are $\mathbb{D}_{\rho=1} = \{x \in \mathbb{R}^2 \mid x^\intercal PB \neq 0\}$ and $\mathbb{D}_{\rho\neq 1} = \{x \in \mathbb{R}^2 \mid x^\intercal PB = 0\}$, respectively. In this example, $\mathbb{D}_{\rho\neq 1}$ is the set of states that are orthogonal to PB.

Although in this simple case, the Lie derivative along g is linear in x, the Lie derivative along g may be an arbitrary nonlinear function of x in general. This example highlights that the inactivity has to be carefully analyzed for nonlinear systems. The inactivity can be evaluated by determining the appropriate Lie derivatives. However, it is generally not possible to analytically determine $\mathbb{D}_{\rho=1}$, except for certain simple cases. Instead, sampling techniques must be used, or a set of nonlinear programs must be solved. Building on the results in [13], we can show that there exist states such that the associated safety filter will be inactive.

Corollary 1: Consider dynamics in (1) with constant $g(x) = B \in \mathbb{R}^{n\times m}$ and a compact set $\mathbb{C} \subseteq \mathbb{D}$ that is parameterized as the zero-superlevel set of a continuously differentiable function $h : \mathbb{D} \mapsto \mathbb{R}$ with $\partial\mathbb{C} = \{x \in \mathbb{D} \mid h(x) = 0\}$. The gradient satisfies $\partial h(x)/\partial x \neq 0$ for all $x \in \partial\mathbb{C}$. For any d-level set $\partial\mathbb{C}_d$ with $d \in (0, h_{\max})$, there exists at least one point $x \in \partial\mathbb{C}_d$, where $\partial h(x)/\partial x \perp B$ such that the associated safety filter in (5) is inactive.

Proof: The proof follows by repeating the argument from [13] for each level set. ■

In our aforementioned example, each level set had exactly two states such that $\mathcal{L}_g h(x) = 0$. In practice, many systems satisfy the condition of a constant $g(x)$, and the requirement of a compact safe set is ubiquitous in the field of safety-critical control. While the result only applies to systems with a constant input matrix, safety filter inactivity can also exist for systems with a state-dependent input matrix $g(x)$.

B. Multiple CBFs to Prevent Inactive Safety Filters

In this section, we propose a method to circumvent inactive safety filters. In particular, we leverage multiple CBFs to achieve a safety filter design that is active for all states x in the safe set. Based on the previous section, one requirement for an active safety filter is that $\mathcal{L}_g h(x) \neq 0$ for all x or, in other words, $\mathbb{D}_{\rho\neq 1} = \emptyset$. We can further require that $\|\mathcal{L}_g h(x)\| \geq \epsilon$, where $\epsilon > 0$ can be chosen such that at least one CBF condition is not redundant. As this may not be satisfied by a single CBF (see Corollary 1), we use a set of $K \in \mathbb{N}$ CBF candidates instead. The safe set is then the positive superlevel set of all CBFs: $\mathbb{C}_{\mathbb{Z}_{1,K}}$. We emphasize that the safe set resulting from intersecting multiple CBFs can be designed such that any desired safe set $\mathbb{X}$ may be approximated. Using this notation, we also define $\mathbb{D}_\epsilon = \mathbb{D} \setminus \bigcap_{i\in\mathbb{Z}_{1,K}} \mathbb{D}_\epsilon^i$ with $\mathbb{D}_\epsilon^i = \{x \in \mathbb{R}^n \mid \|\mathcal{L}_g h_i(x)\| \leq \epsilon\}$ and $\mathbb{U}_{(h_i,\gamma_i)_{i\in\mathbb{Z}_{1,K}}}(x) = \bigcap_{i\in\mathbb{Z}_{1,K}} \mathbb{U}_{(h_i,\gamma_i)}(x)$ with $\mathbb{U}_{(h_i,\gamma_i)}(x) = \{u \in \mathbb{U} \mid \alpha_i(x)u \leq \beta_i(x)\}$.

We formally state how multiple CBFs can prevent the safety filter inactivity issue in the following theorem.

Theorem 3 (ϵ-active CBF safety filter): Consider the system dynamics in (1), a compact control input set $\mathbb{U}$, and an associated set of multiple CBFs $h_i : \mathbb{D} \to \mathbb{R}$, with $i \in \mathbb{Z}_{1,K}$ and $K \in \mathbb{N}$ and a given small positive number $\epsilon > 0$. Let the systems consisting of the system dynamics and each CBF have a least relative degree $\rho_i = 1$. If the intersection of all states for which the norm of the Lie derivative along g is less than or equal to ϵ for each CBF is empty, $\mathbb{D}_\epsilon = \cap_{i\in\mathbb{Z}_{1,K}} \mathbb{D}_\epsilon^i = \emptyset$, then, for all $x \in \mathbb{D}$, it holds that $\max_{i\in\mathbb{Z}_{1,K}} \|\mathcal{L}_g h_i(x)\| > \epsilon$. Furthermore, if

$$\max_{i\in\mathbb{Z}_{1,K}} \|\mathcal{L}_g h_i(x)\| > \bar{\epsilon} := \max_{i\in\mathbb{Z}_{1,K}} \frac{M_{f_i} + \gamma_i(h_{i,\max})}{r - \|u_c\|}$$

where $M_{f_i} > 0$ with $M_{f_i} \geq \|\mathcal{L}_f h_i(x)\|$, and u_c and $r > 0$ are the Chebyshev center and radius, respectively, such that $\mathbb{B}(u_c, r)$ is the largest ball satisfying $\mathbb{B}(u_c, r) \subseteq \mathbb{U}$ and $r - \|u_c\| > 0$, then $\mathbb{U}_{(h_i,\gamma_i)_{i\in\mathbb{Z}_{1,K}}}(x) \subset \mathbb{U}$.

Proof: The result that $\max_{i\in\mathbb{Z}_{1,K}} \|\mathcal{L}_g h_i(x)\| > \epsilon$ follows directly from the definition of the intersection. For each $x \in \mathbb{D}$, we know that, for at least one i, $\mathcal{L}_g h_i(x)$ is nonzero, $\exists i \in \mathbb{Z}_{1,K}, \|\mathcal{L}_g h_i(x)\| > \epsilon$.

Therefore, there exists at least one nonzero half-space constraint such that $\bigcap_{i\in\mathbb{Z}_{1,K}}\{u\in\mathbb{R}^m \mid \alpha_i(x)u \le \beta_i(x)\}\subset\mathbb{R}^m$. As stated at the beginning of the section, to guarantee that the safety filter is always active, $\mathbb{U}_{(h_i,\gamma_i)_{i\in\mathbb{Z}_{1,K}}}(x)\subset\mathbb{U}$, we require for at least one $i\in\mathbb{Z}_{1,K}$ that $\|\mathcal{L}_g h_i(x)\| > \epsilon_i(x)$. We upper bound the right-hand side using $\sigma_{\mathbb{U}}(\alpha_i)\ge\alpha_i^\intercal u_c + r\|\alpha_i\|$ as follows:

$$\epsilon_i(x)\le\frac{M_{f_i}+\gamma_i(h_{i,\max})}{\alpha_i^\intercal(x)u_c+r}\le\frac{(M_{f_i}+\gamma_i(h_{i,\max}))\|\mathcal{L}_g h_i(x)\|}{(r-\|u_c\|)\|\mathcal{L}_g h_i(x)\|}.$$

Since $r-\|u_c\|>0$ and $\|\mathcal{L}_g h_i(x)\|\neq 0$, maximizing over all $i\in\mathbb{Z}_{1,K}$ yields the desired result. ∎

In the above, we have shown how we can use multiple CBFs to prevent safety filter inactivity. In particular, the parameter $\epsilon>0$ trades off conservatism in the feasible inputs and safety filter inactivity. If ϵ is sufficiently large, the safety filter is active for all states $x\in\mathbb{C}$. The derived value for ϵ can lead to conservative or infeasible input sets in practice. The result provides a theoretical upper bound for ϵ that may guide the selection of the K CBFs. Finally, if $\mathbb{D}_\epsilon=\emptyset$ for some $\epsilon>0$, then the associated safety filter (even with $\mathbb{U}=\mathbb{R}^m$) prevents the issue of unbounded $u_{\text{cert},k}$, which was analyzed in [13].

C. Synthesizing Multiple CBFs

In this section, we extend the CBF synthesis problem to multiple CBFs that account for the system's least relative degree. The goal is to synthesize the largest intersection of K zero-superlevel sets $\mathbb{C}_{\{i\}}=\{x\in\mathbb{R}^n|h(x;\theta_i)\ge 0\}$ that are contained inside a desired set of state constraints $\mathbb{X}$. These multiple CBFs are required to satisfy the CBF condition, and the Lie derivative $\mathcal{L}_g h_i(x)$ may never be close to zero. We express this problem using the following optimization problem:

$$\begin{aligned}\max_{\{\theta_i\in\mathbb{R}^p,\phi_i\in\mathbb{R}^q\}_{i=1}^K}\quad & \mathrm{Vol}(\mathbb{C}_{\mathbb{Z}_{1,K}})\\ \text{s.t.}\quad & \mathbb{C}_{\mathbb{Z}_{1,K}}=\bigcap_{i\in\mathbb{Z}_{1,K}}\mathbb{C}_{\{i\}}\subseteq\mathbb{X}\\ & \max_{i\in\mathbb{Z}_{1,K}}\|\mathcal{L}_g h(x;\theta_i)\|\ge\epsilon\quad\forall x\in\mathbb{C}_{\mathbb{Z}_{1,K}}\\ & \bigcap_{i\in\mathbb{Z}_{1,K}}\mathbb{U}_{(h_i,\gamma_i)}(x)\neq\emptyset\quad\forall x\in\mathbb{C}_{\mathbb{Z}_{1,K}}\end{aligned}\tag{8}$$

where θ_i and ϕ_i parameterize h_i and γ_i, respectively, Vol is the set's volume, and $\epsilon>0$ and $K>1$ are user-defined parameters.

The above optimization problem includes multiple semi-infinite constraints, and there is no analytical expression for the volume of the intersection of general superlevel sets. Therefore, we propose to use a sampling-based approach to determine a feasible solution. Our proposed sampling approach may also be implemented as an evolutionary algorithm, which was proposed for synthesizing a single CBF in [27]. To reduce the search space, we sample the function parameters from compact sets Θ and Φ, respectively. For every sampling iteration k, we sample parameters $\theta_{i,k}\in\Theta$ and check if $\mathbb{C}_{\mathbb{Z}_{1,K},k}\subseteq\mathbb{X}$. If $\mathbb{X}$ itself is the zero superlevel set of a continuously differentiable function $h_{\mathbb{X}}(x)$, this can be verified by $\min_{x\in\mathbb{C}_{\mathbb{Z}_{1,K},k}} h_{\mathbb{X}}(x)\ge 0$. Then, for each function $h_{i,k}$, we determine the norm on the Lie derivative and test if $\max_{i\in\mathbb{Z}_{1,K}}\|\mathcal{L}_g h(x;\theta_{i,k})\|\ge\epsilon$ for a sufficiently large set of sampled states $x\in\mathbb{C}_{\mathbb{Z}_{1,K},k}$. Alternatively, we can check the sufficient condition $\min_{x\in\mathbb{C}_{\mathbb{Z}_{1,K},k}}\|\mathcal{L}_g h(x;\theta_{i,k})\|\ge\epsilon$ for all $i\in\mathbb{Z}_{1,K}$ using a nonlinear programming solver. Finally, we sample a parameter $\phi_{i,k_j}\in\Phi$ and need to verify the feasibility of the set of constraints $\mathcal{L}_f h(x,\theta_{i,k})+\mathcal{L}_g h(x;\theta_{i,k})u\ge-\gamma(h(x;\theta_{i,k});\phi_{i,k_j})\ \forall x\in\mathbb{C}_{\mathbb{Z}_{1,K},k}\ \ \forall u\in\mathbb{U}$, and $\forall i\in\mathbb{Z}_{1,K}$. This is a challenging problem for general nonlinear systems and CBFs [6], which often can only be determined by sampling sufficiently many points in $\mathbb{C}_{\mathbb{Z}_{1,K},k}$ or the dynamics and constraints have a special structure that simplifies the computation [28]. In case the feasibility cannot be guaranteed, we can sample a new parameter $\phi_{i,k_{j+1}}\in\Phi$ and execute the feasibility check again. For every successful sample, we approximate the volume of set $\mathbb{C}_{\mathbb{Z}_{1,K}}$, for example, through its bounding box or rejection sampling. Finally, we pick the parameters θ_{i,k^*} and ϕ_{i,k^*}, which lead to the largest surrogate of the true volume of $\mathbb{C}_{\mathbb{Z}_{1,K},k^*}$.

D. Multiple CBFs for Safety Under Sampled-Data Control

In this section, we analyze when a safety filter with multiple CBFs yields safety for discrete-time implementations (i.e., using sampled-data control).

We define $\bar{u}=\max_{u\in\mathbb{U}}\|u\|$ as the maximum feasible control input. A Lipschitz constant for the control affine system's dynamics in (1) can be determined as follows:

$$\|f(x)+g(x)u-f(y)-g(y)u\|\le(L_f+L_g\bar{u})\|x-y\|.$$

We define $L:=(L_f+L_g\bar{u})$. Using this Lipschitz constant, we can bound the difference between the current and future states after applying control input u for sampling time Δt. Consider the following slightly adapted result to explicitly account for the state and control input dependence:

Proposition 1 (Bounded sampled-and-hold deviation [20]): Consider the control-affine system defined in (1). Suppose at time $t_0\ge 0$, a constant control input u is applied to the system for a period of $\Delta t=t-t_0\ge 0$. The distance between the future state $x(t)$ and the state $x_{t_0}=x(t_0)$ is $e(t,x_{t_0},u)=\|x(t)-x_{t_0}\|$. Let $\bar{e}(t,x_{t_0},u):=\frac{1}{L}\|f(x_{t_0})+g(x_{t_0})u\|(\exp(L\Delta t)-1)$. Then, $e(t,x_{t_0},u)\le\bar{e}(t,x_{t_0},u)$ for all $t\ge t_0$.

Proof: The proof follows from [20]. ∎

With Proposition 1 and compact $\mathbb{U}$, we can show that trajectories starting in $\mathbb{C}_{\mathbb{Z}_{1,k}}$ will stay bounded.

Lemma 1 (Bounded trajectories): The error $e(t,x,u)$ is upper bounded for any $x\in\mathbb{C}_{\mathbb{Z}_{1,k}}$ and any $u\in\mathbb{U}$ with

$$\bar{e}(t,x,u)\le\frac{\bar{f}+\bar{g}\bar{u}}{L}(\exp(L\Delta t)-1)=:\hat{e}(\Delta t)\tag{9}$$

where $\bar{f}=\max_{x\in\mathbb{C}_{\mathbb{Z}_{1,k}}}\|f(x)\|$ and $\bar{g}=\max_{x\in\mathbb{C}_{\mathbb{Z}_{1,k}}}\|g(x)\|$.

Proof: The dynamics are upper bounded by $\|f(x)+g(x)u\|\le\bar{f}+\bar{g}\bar{u}$. The maxima $\bar{f}$ and $\bar{g}$ exist because f and g are Lipschitz continuous on the bounded set $\mathbb{C}_{\mathbb{Z}_{1,k}}$. Finally, the maxima upper bound the deviation for all states $x\in\mathbb{C}_{\mathbb{Z}_{1,k}}$. This gives the desired result. ∎

We leverage the result in Lemma 1 to determine a positive sampling time Δt that yields control invariance for the sampled-data control system. This is formalized in the following theorem.

Theorem 4 (Safe sampled-data control): Consider the sample-and-hold system in (6). Let $\mathbb{U}$ be a compact set, and let the sample-and-hold error for each CBF be bounded by $e_{i,\max}=-\frac{\gamma_i(-d_i)}{L_\pi M_i}$. If the sampling time $\Delta t>0$ and

$$\Delta t\le\frac{1}{L}\ln\left(1+\frac{\min_{i\in\mathbb{Z}_{1,K}}(-\gamma_i(-d_i))\,L}{L_\pi M_i(\bar{f}+\bar{g}\bar{u})}\right)\tag{10}$$

for $d_i\in(0,h_{i,\max})$ such that $\mathbb{C}_d=\bigcap_{i\in\mathbb{Z}_{1,K}}\mathbb{C}_{d_i}=\{x\in\mathbb{R}^n\mid h_i(x)-d_i\ge 0\}\neq\emptyset$ for each $i\in\mathbb{Z}_{1,K}$, then the sample-and-hold system (6) can be rendered positively control invariant on $\mathbb{C}_{\mathbb{Z}_{1,K}}$ using Theorem 2 if the resulting control input set $\mathbb{U}_{(h_i,\gamma_i)_{i\in\mathbb{Z}_{1,K}}}(x)\neq\emptyset\ \forall x\in\mathbb{C}_{\mathbb{Z}_{1,K}}$.

Proof: As the upper bound on the sampled-data error $\hat{e}(\Delta t_i)$ is bounded by $e_{i,\max}$ we get

$$\frac{\bar{f}+\bar{g}\bar{u}}{L}\left(\exp(L\Delta t_i)-1\right) \leq -\frac{\gamma_i(-d_i)}{L_\pi M_i}.$$

Due to Proposition 1 and Lemma 1, satisfying this inequality yields a valid bound on all feasible $e(\Delta t_i, x, u)$ with $x \in \mathbb{C}_{\mathbb{Z}_{1,K}}$ and $u \in \mathbb{U}_{(h_i,\gamma_i)_{i\in\mathbb{Z}_{1,K}}}(x)$. Then, solving for Δt_i results in

$$\Delta t_i \leq \frac{1}{L}\ln\left(1+\frac{-\gamma_i(-d_i)L}{L_\pi M_i(\bar{f}+\bar{g}\bar{u})}\right).$$

Taking the minimum over all $i \in \mathbb{Z}_{1,K}$ and moving the minimum into the logarithm gives the desired result. Finally, the last condition on the resulting control input set guarantees that the resulting CBF condition yields a feasible control input at every state inside the safe set. ■

While this lets us compute the required sampling time Δt, in practice, it is usually fixed by the system or hardware. Thus, given Δt, we want to check if it ensures safety by determining the necessary tightening d_i for each CBF, as stated in the following corollary.

Corollary 2: Let $\mathbb{U}$ be a compact set and let the sample-and-hold error for each CBF be bounded by $e_{i,\max} = -\frac{\gamma_i(-d_i)}{L_\pi M_i}$. If $d_i \in (0, h_{i,\max})$ for each $i \in \mathbb{Z}_{1,K}$ satisfies

$$d_i \geq -\gamma_i^{-1}\left(\frac{L_\pi M_i(\bar{f}+\bar{g}\bar{u})}{L}\left(1-\exp(L)\Delta t)\right)\right)$$

with $\Delta t > 0$ such that $\mathbb{C}_d = \bigcap_{i\in\mathbb{Z}_{1,K}} \mathbb{C}_{d_i} = \{x \in \mathbb{R}^n \mid h_i(x) - d_i \geq 0\} \neq \emptyset$. Then, the sample-and-hold system (6) can be rendered positively control invariant on $\mathbb{C}_{\mathbb{Z}_{1,k}}$ if the resulting control input set $\mathbb{U}_{(h_i,\gamma_i)_{i\in\mathbb{Z}_{1,K}}}(x) \neq \emptyset\ \forall x \in \mathbb{C}_{\mathbb{Z}_{1,K}}$.

VI. Evaluation

A. Simulation Example

We verify our theoretical results in simulation. Details of the simulation setup, along with implementations, can be found online.[1]

We consider a continuous-time double integrator dynamics model $\dot{x} = Ax + Bu$ with $x \in \mathbb{R}^2$ and $u \in \mathbb{R}$. We investigate two cases: 1) using a single CBF (least relative degree is one) and 2) using two CBFs (joint relative degree is one). In the first case, we use the CBF $h(x) = 1-(x-c)^\intercal P(x-c)$, where $c = \begin{bmatrix}0 & 0\end{bmatrix}^\intercal$ and $P = \operatorname{diag}(1,2)$. For the second case, we solve the optimization in (8) over $\theta_i = \begin{bmatrix}\operatorname{diag}(P_i)^\intercal & c_i^\intercal\end{bmatrix}^\intercal$ and a fixed slope of a linear class-$\mathcal{K}_e$ function expressed as $\phi_i = 2$ such that $\epsilon = 0.01$ and $\mathbb{C}_{\{1,2\}} \subseteq \mathbb{C} = \mathbb{X}$. This yields a safety filter with the CBFs $h_i(x) = 1-(x-c_i)^\intercal P_i(x-c_i)$, with $i \in \{1,2\}$, where $c_1 = \begin{bmatrix}0.01 & -1.27\end{bmatrix}^\intercal$, $c_2 = \begin{bmatrix}0.02 & 1.33\end{bmatrix}^\intercal$, $P_1 = \operatorname{diag}(0.60, 0.26)$, and $P_2 = \operatorname{diag}(0.56, 0.26)$. For the single CBF, we also use $\gamma(r) = \gamma_i(r) = 2r$. The unsafe control input policy is $\pi(x) = -1.0$, and the input constraint set is $\mathbb{U} = \{u \in \mathbb{R} \mid |u| \leq 2.0\}$. For both cases, we use a control frequency of 10 kHz in the simulation. We use Theorem 4 to determine the required tightenings $d > 0$ and $d_i > 0\ \forall i \in \{1,2\}$ for each CBF to apply our results for sampled data systems. Our calculations yield $d_1 = 0.0002$ and $d_2 = 0.0007$. The high control frequency leads to a small tightening of the safe sets. The chattering for the single-CBF approach yields a large Lipschitz constant for the closed-loop policy, such that no feasible tightening exists.

[1] [Online]. Available: https://github.com/lukasbrunke/multi-cbf

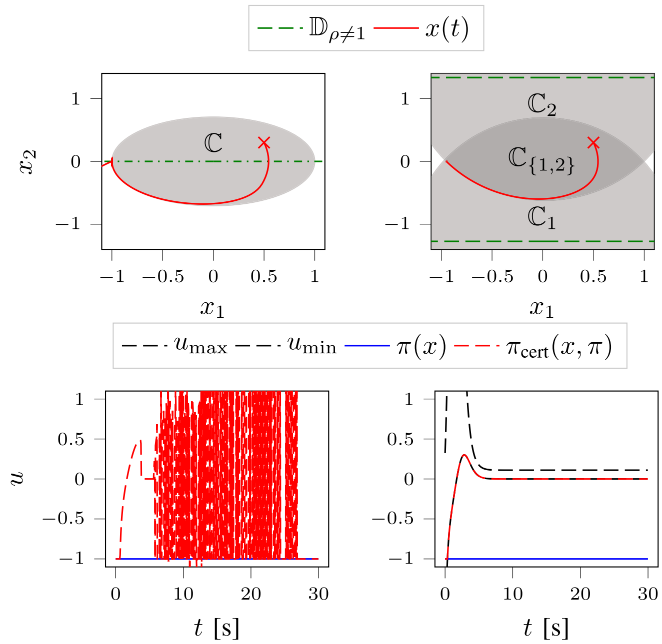

Fig. 3. Simulation comparing single-CBF safety filter with chattering (left) versus our multi-CBF safety filter preventing high-frequency control (right). The states with relative degree $\rho \neq 1$ are shown in green. Both filters use uncertified policy $\pi(x) = -1.0$. The single-CBF case exhibits constraint set violations and input chattering due to loose constraints near $x \in \mathbb{D}_{\rho\neq 1}$ (not shown to avoid clutter). Our multi-CBF approach ensures safety filter activity and prevents chattering.

Instead, we use a nominal implementation for the single-CBF case. In Fig. 3, we see that the single-CBF safety filter results in chattering and safe set violations (left) and that our proposed multi-CBF safety filter successfully prevents a high-frequency control input signal (right). The multiple CBFs are tightened according to our theoretical results in Theorem 4 to achieve a safe closed-loop sampled-data system. We highlight the states where the relative degree $\rho \neq 1$ in green (top). Note that these states are shown using a dotted-dashed line for the single CBF case and a dashed line for each CBF in the case with multiple CBFs. However, since the intersection of these sets is empty in the multiple CBF case, no states satisfy $\rho \neq 1$ for both CBFs simultaneously. This prevents the safety filter with the multiple CBFs from being inactive. For both safety filters, we apply an uncertified control input policy (blue line in the bottom plots). While our proposed approach never violates the safety constraints, the single-CBF approach violates the constraints (red line in the top plots) and yields a chattering behavior as visible in the input trajectory (bottom plot). This is because the single CBF safety filter becomes inactive close to states $x \in \mathbb{X}_{\rho\neq 1}$. For the safety filter with multiple CBFs, there is no chattering. We also highlight the input constraints (black dashed lines) and show that the lower bound on the constraint converges to 0. Therefore, no chattering is possible with our proposed safety filter.

Solving QPs at high rates (e.g., 10 kHz) is often impractical on resource-constrained hardware. We therefore repeated the simulation at 100 Hz, for which theoretical guarantees no longer hold, as tightened safe sets lead to infeasible QPs. However, similar behavior was observed without tightening, and results (not shown) closely match those in Fig. 3. While theoretical conditions are not met, using multiple CBFs still improves closed-loop performance over a single CBF. This suggests that lower control frequencies may suffice in practice.

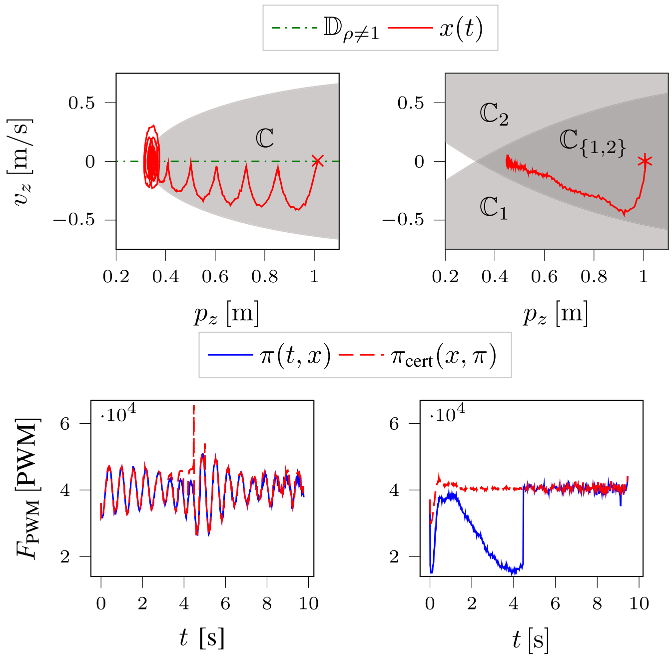

Fig. 4. Real-world quadrotor comparison: single-CBF safety filter violating the safe set (left) versus our multi-CBF safety filter preventing violations (right). Both use 60 Hz control frequency with identical uncertified tracking policy $\pi(t,x)$. The multi-CBF filter achieves safety, while the single-CBF case exhibits oscillatory behavior and safe set violations. The states with relative degree $\rho \neq 1$ are shown in green.

B. Quadrotor Experiments

We verify our proposed method of multiple CBFs to prevent inactive safety filters on a real-world nonlinear quadrotor system. A picture of the experiment is shown in Fig. 1, and a video can be found online.[2]

We leverage the quadrotor's roll, pitch, yaw, and collective thrust interface, where the thrust is commanded through pulsewidth modulation. As the y-position is controlled to be 0, the reduced system state is $x = [p_x \;\; p_z \;\; \theta \;\; v_x \;\; v_z]^\intercal \in \mathbb{R}^5$, where θ is the quadrotor's pitch angle, and the commanded control input is $u = [\theta_{\text{desired}} \;\; F_{\text{desired}}]^\intercal \in \mathbb{R}^2$. The identified nonlinear model is $\dot{x} = f(x) + g(x)u$, where $f(x) = [v_x \;\; v_z \;\; \alpha_1\theta \;\; \beta_1\sin(\theta) \;\; \beta_1\cos(\theta) - g]^\intercal$, $g(x) = \operatorname{diag}(g_1, g_2(x))$, $g_1 = [0 \;\; 0 \;\; \alpha_2]^\intercal$, $g_2(x) = \beta_2[\sin(\theta) \;\; \cos(\theta)]^\intercal$, $\alpha_1 = -\alpha_2 = -60.00$, $\beta_1 = 4.60$, and $\beta_2 = 15.40$. We use a position controller $\pi(t,x)$ to track an unsafe reference trajectory. In the first case, we use a single CBF $h(x) = 1 - (x-c)^\intercal P(x-c)$, where $c = [0.0 \;\; 1.52 \;\; 0.0 \;\; 0.0 \;\; 0.0]^\intercal$ and $P = \operatorname{diag}(0.0, 0.7, 0.0, 0.0, 2.0)$. In the second case, we apply a safety filter with the CBFs $h_i(x) = 1 - (x-c_i)^\intercal P_i(x-c_i)$, with $i \in \{1,2\}$, where $c_1 = [0.0 \;\; 1.55 \;\; 0.0 \;\; 0.0 \;\; 2.5]^\intercal$, $c_2 = [0.0 \;\; 1.55 \;\; 0.0 \;\; 0.0 \;\; -2.5]^\intercal$, and $P_1 = P_2 = \operatorname{diag}(0.0, 0.25, 0.0, 0.0, 0.1)$. The CBFs are chosen such that $\mathbb{C}_{\{1,2\}} \subseteq \mathbb{C} = \mathbb{X}$. For all CBFs, we use $\gamma(r) = \gamma_i(r) = 3r$, and we implement the safety filters at a frequency of 60 Hz.

For both safety filters, we initialize the system at approximately $x = [0.0 \;\; 1.0 \;\; 0.0 \;\; 0.0 \;\; 0.0]^\intercal$ and apply the same uncertified policy $\pi(t,x)$ (blue line in Fig. 4). We show that the safety filter with multiple CBFs (plots on the right-hand side in Fig. 4) achieves safety. In contrast, the safety filter with a single CBF (plots on the left-hand side) violates the safe set. In addition, it yields an oscillatory behavior in the certified control input and the closed-loop state trajectory. We highlight the states where $\mathcal{L}_g h(x) = 0$ in green. These states are not visible in the state space plot on the right (the case with multiple CBFs). However, since the intersection of these sets is empty in the multiple CBF case, no states satisfy $\mathcal{L}_g h_1(x) = 0 \wedge \mathcal{L}_g h_2(x) = 0$ for both CBFs simultaneously. This prevents the safety filter with the multiple CBFs from being inactive and keeps the system inside the safe set $\mathbb{C}_{\{1,2\}}$. Due to the choice of the class $\mathcal{K}_e$ function, the quadrotor stays further away from the boundary of the safe set compared to the system in the simulation result; see Fig. 1.

[2] [Online]. Available: http://tiny.cc/multi-cbf

VII. Conclusion

In this work, we investigated the issue of incorrect relative degree assumptions, how they result in inactive CBF safety filters, and their negative impact on discrete-time implementations, such as chattering and safe set violations. We characterize when a CBF safety filter is inactive, analyze the direction and magnitudes of control inputs at states of inactivity, and discuss the implications of the discrete-time implementation. We present a method that leverages multiple CBFs to prevent inactive safety filters and an upper bound on the sampling time to ensure safety in discrete-time implementations. From the simulation and quadrotor experiments, we verify and show that our proposed method efficiently mitigates chattering and safe set violations caused by inactive safety filters.

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