```
In[1]:= Remove["Global`*"]
```

Sensitivity of Joint Estimation in Multi-Agent Iterative Learning Control

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Note that we use the Latin letters *a, b, d, e, eNom, g, L, p11,* and *p11Perfect* to represent α, β, δ, ϵ, $\bar{\epsilon}$, γ, λ, $p_j^{(1,1)}$, and ${}^*p_j^{(1,1)}$, respectively, in the formulas below.

Section 4.1: Variance of Disturbance Estimate

Variance of the disturbance estimate in the ideal case assuming $\epsilon = \bar{\epsilon}$, expressed in terms of γ and ϵ.

```
In[2]:= p11Perfect = FullSimplify[(a + b + j b^2 + j N a b) / ((1 + j b) (1 + j b + j N a)) /. {a → e g, b → (1 - e) g}]
```

$$\text{Out[2]=}\quad \frac{g\,(-1 + (-1 + e)\,g\,j\,(1 + e\,(-1 + N)))}{(-1 + (-1 + e)\,g\,j)\,(1 + g\,j\,(1 + e\,(-1 + N)))}$$

Proposition 1:

Variance of the disturbance estimate in the general case $\epsilon \neq \bar{\epsilon}$. (This expression is derived and proven in the Appendix, at the bottom of this document.)

```
In[3]:= p11 =
  FullSimplify[FactorTerms[(g (1 + (3 j + ((N - 2) * 2 j - (N - 1) 2 j d) eNom - j (N - 1) eNom^2) g - j^2
          (-3 + 4 (-(N - 2) + (N - 1) / 2 d) eNom + (-(N - 3)^2 + N + 2 + (N - 1) (N - 2) d) eNom^2 +
            (N - 1) (N - 2) eNom^3) g^2 + j^3 (1 + (N - 2) eNom - (N - 1) eNom^2)^2 g^3)) /
     ((1 - j (-1 + eNom) g)^2 (1 + (j + j (N - 1) eNom) g)^2)]]
```

$$\text{Out[3]=}\quad \Big(g\,((-1 + (-1 + eNom)\,g\,j)\,(1 + g\,j\,(1 + eNom\,(-1 + N)))\,(-1 + (-1 + eNom)\,g\,j\,(1 + eNom\,(-1 + N))) - d\,eNom\,g\,j\,(2 + g\,j\,(2 + eNom\,(-2 + N)))\,(-1 + N))\Big) \Big/ \Big((-1 + (-1 + eNom)\,g\,j)^2\,(1 + g\,j\,(1 + eNom\,(-1 + N)))^2\Big)$$

Dividing $p_j^{(1,1)}$ into $f_j(\gamma, \bar{\epsilon}, \text{N})$ and $g_j(\gamma, \bar{\epsilon}, \text{N})$.

```
In[4]:= fj = FullSimplify[p11 /. {d → 0}]
```

$$\text{Out[4]=}\quad \frac{g\,(-1 + (-1 + eNom)\,g\,j\,(1 + eNom\,(-1 + N)))}{(-1 + (-1 + eNom)\,g\,j)\,(1 + g\,j\,(1 + eNom\,(-1 + N)))}$$

```
In[5]:= gj = FullSimplify[D[p11, d]] * (-1)
```

$$\text{Out[5]=}\quad \frac{eNom\,g^2\,j\,(2 + g\,j\,(2 + eNom\,(-2 + N)))\,(-1 + N)}{(-1 + (-1 + eNom)\,g\,j)^2\,(1 + g\,j\,(1 + eNom\,(-1 + N)))^2}$$

```
In[6]:= check = FullSimplify[p11 - (fj - d * gj)]

Out[6]= 0
```

Independent Estimation with N=1 results in equal variances for both cases, the perfect knowledge case with $\epsilon = \bar{\epsilon}$ and the general case $\epsilon \neq \bar{\epsilon}$.

```
In[7]:= p11PerfectIndep = FullSimplify[p11Perfect /. {N → 1}]
```

$$\text{Out[7]=}\quad \frac{g}{1 + g\,j}$$

In[8]:=
```
p11Indep = FullSimplify[p11 /. {N → 1}]
```

Out[8]= $\frac{g}{1+g\,j}$

with

In[9]:=
```
FullSimplify[D[p11Indep , j]]
```

Out[9]= $-\frac{g^2}{(1+g\,j)^2}$

Joint Estimation shows a larger variance in the case of $\epsilon \neq \bar{\epsilon}$.

In[10]:=
```
diff = FullSimplify[p11 - p11Perfect /. e → (eNom + d)]
```

Out[10]= $-\left(d^2\,g^2\,j\,(1+g\,j\,(3+g\,j\,(3+g\,j+\text{eNom}\,(d\,(2+g\,j\,(2+\text{eNom}\,(-2+N)))+\text{eNom}\,(3+g\,j\,(3+\text{eNom}\,(-2+N))))\,(-1+N))))\,(-1+N)\right)\Big/\left((-1+(-1+\text{eNom})\,g\,j)^2\,(-1+(-1+d+\text{eNom})\,g\,j)\,(1+g\,j\,(1+\text{eNom}\,(-1+N)))^2\,(1-(-1+d+\text{eNom})\,g\,j+(d+\text{eNom})\,g\,j\,N)\right)$

The difference is positive for δ, γ, $\bar{\epsilon}$, j > 0, N>1, and $\delta + \bar{\epsilon} \leqslant 1$.

-- Partial derivative of $p_j^{(1,1)}$ with respect to δ and ϵ .

In[11]:=
```
FullSimplify[D[p11, d]]
```

Out[11]= $-\frac{\text{eNom}\,g^2\,j\,(2+g\,j\,(2+\text{eNom}\,(-2+N)))\,(-1+N)}{(-1+(-1+\text{eNom})\,g\,j)^2\,(1+g\,j\,(1+\text{eNom}\,(-1+N)))^2}$

In[12]:=
```
FullSimplify[D[p11 /. d → e - eNom, e]]
```

Out[12]= $-\frac{\text{eNom}\,g^2\,j\,(2+g\,j\,(2+\text{eNom}\,(-2+N)))\,(-1+N)}{(-1+(-1+\text{eNom})\,g\,j)^2\,(1+g\,j\,(1+\text{eNom}\,(-1+N)))^2}$

-- Limit behavior of $f_j(\gamma, \bar{\epsilon}, N)$ for j → ∞ .

In[13]:=
```
FullSimplify[D[fj, j]]
```

Out[13]= $\left(g^2\,\left(-1+\text{eNom}^2-2\,(-1+\text{eNom})^2\,g\,j-(-1+\text{eNom})^4\,g^2\,j^2+\text{eNom}\,\left(-\text{eNom}+2\,(-1+\text{eNom})\,g\,j+2\,(-1+\text{eNom})^3\,g^2\,j^2\right)\,N-(-1+\text{eNom})^2\,\text{eNom}^2\,g^2\,j^2\,N^2\right)\right)\Big/\left((-1+(-1+\text{eNom})\,g\,j)^2\,(1+g\,j\,(1+\text{eNom}\,(-1+N)))^2\right)$

In[14]:=
```
FullSimplify[Limit[fj, j → Infinity]]
```

Out[14]= 0

-- Limit behavior of $p_j^{(1,1)}$ for j → ∞ and $\bar{\epsilon} \neq 1$.

In[15]:=
```
FullSimplify[Limit[p11, j → Infinity]]
```

Out[15]= 0

-- Limit of $p_j^{(1,1)}$ for j → ∞ and $\bar{\epsilon} = 1$, denoted by $\ell(\gamma, \delta, N)$.

*** Note that in this case, δ is smaller than zero, -1 ⩽ δ ⩽ 0. ***

In[16]:=
```
ell = FullSimplify[Limit[p11 /. eNom → 1, j → Infinity]]
```

Out[16]= $d\,g\,\left(-1+\frac{1}{N}\right)$

In[17]:=
```
FullSimplify[D[ell, N]]
```

Out[17]= $-\frac{d\,g}{N^2}$

In[18]:= `FullSimplify[D[ell, d]]`

Out[18]= $$g\left(-1+\frac{1}{N}\right)$$

In[19]:= `FullSimplify[D[ell, g]]`

Out[19]= $$d\left(-1+\frac{1}{N}\right)$$

In[20]:= `FullSimplify[Limit[ell /. d → -1, N → Infinity]]`

Out[20]= g

Section 4.2: Performance Index

Performance index $^{*}R$ in the ideal case assuming $\epsilon = \bar{\epsilon}$.

In[21]:= `Rperfect = FullSimplify[Together[(p11PerfectIndep + L) / (p11Perfect + L)]]`

Out[21]= $$-\frac{(-1+(-1+e)\,g\,j)\,(g+L+g\,j\,L)\,(1+g\,j\,(1+e\,(-1+N)))}{(1+g\,j)\,(L+g\,(1+j\,L\,(2+e\,(-2+N)))-(-1+e)\,g\,j\,(1+j\,L)\,(1+e\,(-1+N))))}$$

In[22]:= `FullSimplify[Limit[Rperfect /. {e → 1, L → 0}, j → Infinity]]`

Out[22]= N

In[23]:= `FullSimplify[Limit[Rperfect, j → Infinity]]`

Out[23]= 1

Performance index R in the general case assuming $\epsilon \neq \bar{\epsilon}$.

In[24]:= `R = FullSimplify[(p11Indep + L) / (p11 + L)]`

Out[24]= $$\frac{\frac{g}{1+g\,j}+L}{L+\frac{g\,((-1+(-1+eNom)\,g\,j)\,(1+g\,j\,(1+eNom\,(-1+N)))\,(-1+(-1+eNom)\,g\,j\,(1+eNom\,(-1+N)))-d\,eNom\,g\,j\,(2+g\,j\,(2+eNom\,(-2+N)))\,(-1+N))}{(-1+(-1+eNom)\,g\,j)^2\,(1+g\,j\,(1+eNom\,(-1+N)))^2}}$$

Lemma 2:

-- Limit behavior of R for $j \to \infty$ and $\bar{\epsilon} \neq 1$.

In[25]:= `FullSimplify[Limit[R, j → Infinity]]`

Out[25]= 1

-- Limit of R for $j \to \infty$ and $\bar{\epsilon} = 1$, denoted by $\ell_R(\gamma, N, \delta, \lambda)$.

*** Note that in this case, δ is smaller than zero, $-1 \leqslant \delta \leqslant 0$. ***

In[26]:= `ellR = FullSimplify[Limit[R /. eNom → 1, j → Infinity]]`

Out[26]= $$\frac{L\,N}{d\,g-d\,g\,N+L\,N}$$

In[27]:= `FullSimplify[D[ellR , N]]`

Out[27]= $$\frac{d\,g\,L}{(d\,g\,(-1+N)-L\,N)^2}$$

In[28]:= `FullSimplify[D[ellR , d]]`

Out[28]= $$\frac{g\,L\,(-1+N)\,N}{(L\,N+d\,(g-g\,N))^2}$$

In[29]:= `FullSimplify[Limit[ellR /. d → -1, N → Infinity]]`

Out[29]= $$\frac{L}{g+L}$$

Lemma 3:
The non-positive partial derivative below allows us to state the implication (59).

In[30]:= `FullSimplify[D[fj , N]]`

Out[30]= $$\frac{\text{eNom}^2\, g^2\, j}{(-1+(-1+\text{eNom})\, g\, j)\, (1+g\, j\, (1+\text{eNom}\, (-1+N)))^2}$$

Partial derivative of R with respect to $\bar{\epsilon}$.

In[31]:= `FullSimplify[D[R /. d → e - eNom, eNom]]`

Out[31]= $$-\Bigg(2\,(e-\text{eNom})\, g^2\, j\left(\frac{g}{1+g\,j}+L\right)\left(1+g\,j\left(3+g\,j\left(3+g\,j+\text{eNom}^2\,(3+g\,j\,(3+\text{eNom}\,(-2+N)))\,(-1+N)\right)\right)\right)(-1+N)\Bigg)\Big/\Big((-1+(-1+\text{eNom})\,g\,j)^3\,(1+g\,j\,(1+\text{eNom}\,(-1+N)))^3\Big(L+(g\,((-1+(-1+\text{eNom})\,g\,j)\,(1+g\,j\,(1+\text{eNom}\,(-1+N)))\,(-1+(-1+\text{eNom})\,g\,j\,(1+\text{eNom}\,(-1+N)))-(e-\text{eNom})\,\text{eNom}\,g\,j\,(2+g\,j\,(2+\text{eNom}\,(-2+N)))\,(-1+N)))\Big/\Big((-1+(-1+\text{eNom})\,g\,j)^2\,(1+g\,j\,(1+\text{eNom}\,(-1+N)))^2\Big)\Big)^2\Big)$$

Lemma 4:
To derive the inequality (61), we use $\delta = \epsilon - \bar{\epsilon}$. The right side of the inequality is

In[32]:= `IneqR = FullSimplify[(fj - (fj /. N → 1) + eNom gj) / gj]`

Out[32]= $$\frac{\text{eNom}\left(1+g\,j\left(2+g\,j\left(1+\text{eNom}^2\,(-1+N)\right)\right)\right)}{(1+g\,j)\,(2+g\,j\,(2+\text{eNom}\,(-2+N)))}$$

and therefore $h_j(\gamma,\ \bar{\epsilon}, N)$

In[33]:= `hj = IneqR / eNom`

Out[33]= $$\frac{1+g\,j\left(2+g\,j\left(1+\text{eNom}^2\,(-1+N)\right)\right)}{(1+g\,j)\,(2+g\,j\,(2+\text{eNom}\,(-2+N)))}$$

which is smaller than 1

In[34]:= `FullSimplify[Together[1 - hj]]`

Out[34]= $$-\frac{(-1+(-1+\text{eNom})\,g\,j)\,(1+g\,j\,(1+\text{eNom}\,(-1+N)))}{(1+g\,j)\,(2+g\,j\,(2+\text{eNom}\,(-2+N)))}$$

and

In[35]:= `FullSimplify[D[hj, N]]`

Out[35]= $$-\frac{\text{eNom}\,g\,j\,(-1+(-1+\text{eNom})\,g\,j)^2}{(1+g\,j)\,(2+g\,j\,(2+\text{eNom}\,(-2+N)))^2}$$

In[36]:= `hjLimN = FullSimplify[Limit[hj, N → Infinity]]`

Out[36]= $$\frac{\text{eNom}\,g\,j}{1+g\,j}$$

In[37]:= `FullSimplify[Limit[hjLimN , j → Infinity]]`

Out[37]= eNom

In[38]:=

```
Remove[p11]
```

Appendix A: Proof of Proposition 1

The covariance matrix is defined by four elements $p_j^{(0,0)}$, $p_j^{(0,1)}$, $p_j^{(1,1)}$, $p_j^{(1,2)}$. These are denoted by *p00[j_], p01[j_], p11[j_], p12[j_]* below.

(I) Explicit representation of the Kalman gains K_j.

First, we derive an explicit representation of the Kalman gains K_j. The Kalman gain is defined by the three scalar values $k_j^{(0,1)}$, $k_j^{(1,1)}$, $k_j^{(1,2)}$ denoted by *k01j, k11j, k12j* below. In Schöllig et al. (2010), the Kalman gains were given in terms of the matrix S_j, see Section 3, eq.(19), which is analogously defined by four scalar values $s_j^{(0,0)}$, $s_j^{(0,1)}$, $s_j^{(1,1)}$, $s_j^{(1,2)}$. From Schöllig et al. (2010) eq. (43) with the nominal distrubance variances $\overline{\alpha}$, $\overline{\beta}$ (denoted by *aNom, bNom*)

In[39]:=
```
s00[j_] = FullSimplify[(1 + j bNom) aNom / (1 + j bNom + j N aNom) /. {aNom → eNom g, bNom → (1 - eNom) g}]
```

Out[39]= $$\frac{\text{eNom } g\ (1 + g\ (j - \text{eNom } j))}{1 + g\ j\ (1 + \text{eNom}\ (-1 + N))}$$

In[40]:=
```
s01[j_] = FullSimplify[aNom / (1 + j bNom + j N aNom) /. {aNom → eNom g, bNom → (1 - eNom) g}]
```

Out[40]= $$\frac{\text{eNom } g}{1 + g\ j\ (1 + \text{eNom}\ (-1 + N))}$$

In[41]:=
```
s11[j_] = FullSimplify[(aNom + bNom + j bNom^2 + j N aNom bNom) / (1 + j bNom) / (1 + j bNom + j N aNom) /.
   {aNom → eNom g, bNom → (1 - eNom) g}]
```

Out[41]= $$\frac{g\ (-1 + (-1 + \text{eNom})\ g\ j\ (1 + \text{eNom}\ (-1 + N)))}{(-1 + (-1 + \text{eNom})\ g\ j)\ (1 + g\ j\ (1 + \text{eNom}\ (-1 + N)))}$$

In[42]:=
```
s12[j_] = FullSimplify[aNom / (1 + j bNom) / (1 + j bNom + j N aNom) /. {aNom → eNom g, bNom → (1 - eNom) g}]
```

Out[42]= $$-\frac{\text{eNom } g}{(-1 + (-1 + \text{eNom})\ g\ j)\ (1 + g\ j\ (1 + \text{eNom}\ (-1 + N)))}$$

From Schöllig et al. (2010) eq. (41) with (38) and (39), we obtain the closed-form representation of the Kalman gains.

In[43]:=
```
n1 = FullSimplify[1 + s11[j - 1] - s12[j - 1]]
```

Out[43]= $$\frac{-1 + (-1 + \text{eNom})\ g\ j}{-1 + (-1 + \text{eNom})\ g\ (-1 + j)}$$

In[44]:=
```
n2 = FullSimplify[1 + s11[j - 1] + (N - 1) s12[j - 1]]
```

Out[44]= $$\frac{1 + g\ j\ (1 + \text{eNom}\ (-1 + N))}{1 + g\ (-1 + j)\ (1 + \text{eNom}\ (-1 + N))}$$

In[45]:=
```
m11 = FullSimplify[(1 + s11[j - 1] + (N - 2) s12[j - 1]) / n1 / n2]
```

Out[45]= $$\frac{-1 + g\ (1 - j\ (2 + \text{eNom}\ (-2 + N))) + (-1 + \text{eNom})\ g^2\ (-1 + j)\ j\ (1 + \text{eNom}\ (-1 + N))}{(-1 + (-1 + \text{eNom})\ g\ j)\ (1 + g\ j\ (1 + \text{eNom}\ (-1 + N)))}$$

In[46]:=
```
m12 = FullSimplify[-s12[j - 1] / n1 / n2]
```

Out[46]= $$\frac{\text{eNom } g}{(-1 + (-1 + \text{eNom})\ g\ j)\ (1 + g\ j\ (1 + \text{eNom}\ (-1 + N)))}$$

In[47]:= `k01j = FullSimplify[s01[j - 1] (m11 + (N - 1) m12)]`

Out[47]=
$$\frac{\text{eNom } g}{1 + g\, j\, (1 + \text{eNom}\, (-1 + N))}$$

In[48]:= `k12j = FullSimplify[s11[j - 1] m12 + s12[j - 1] m11 + (N - 2) s12[j - 1] m12]`

Out[48]=
$$-\frac{\text{eNom } g}{(-1 + (-1 + \text{eNom})\, g\, j)\, (1 + g\, j\, (1 + \text{eNom}\, (-1 + N)))}$$

In[49]:= `k11j = FullSimplify[s11[j - 1] m11 + (N - 1) s12[j - 1] m12]`

Out[49]=
$$\frac{g\, (-1 + (-1 + \text{eNom})\, g\, j\, (1 + \text{eNom}\, (-1 + N)))}{(-1 + (-1 + \text{eNom})\, g\, j)\, (1 + g\, j\, (1 + \text{eNom}\, (-1 + N)))}$$

In[50]:= `check = FullSimplify[k11j - fj]`

Out[50]= `0`

(II) Recursive equations for $p_j^{(0,0)}$, $p_j^{(0,1)}$, $p_j^{(1,1)}$, $p_j^{(1,2)}$

Second, from eq. (22) with the closed form of the Kalman gain K_j derived above, recursive equations are derived for $p_j^{(0,0)}$, $p_j^{(0,1)}$, $p_j^{(1,1)}$, $p_j^{(1,2)}$. Matrices of intermediate results are symmetric and represented by its scalar entries similar to how it was done for S_j.

Compute A = (I - K_j H) P_{j-1}.

In[51]:= `a00 = FullSimplify[p00[j - 1] - N k01j p01[j - 1]]`

Out[51]=
$$\text{p00}[-1 + j] - \frac{\text{eNom } g\, N\, \text{p01}[-1 + j]}{1 + g\, j\, (1 + \text{eNom}\, (-1 + N))}$$

In[52]:= `a01 = FullSimplify[p01[j - 1] - p11[j - 1] k01j - (N - 1) p12[j - 1] k01j]`

Out[52]=
$$\frac{(1 + g\, j\, (1 + \text{eNom}\, (-1 + N)))\, \text{p01}[-1 + j] - \text{eNom } g\, (\text{p11}[-1 + j] + (-1 + N)\, \text{p12}[-1 + j])}{1 + g\, j\, (1 + \text{eNom}\, (-1 + N))}$$

In[53]:= `a10 = FullSimplify[(1 - k11j - (N - 1) k12j) p01[j - 1]]`

Out[53]=
$$\frac{(1 + g\, (-1 + j)\, (1 + \text{eNom}\, (-1 + N)))\, \text{p01}[-1 + j]}{1 + g\, j\, (1 + \text{eNom}\, (-1 + N))}$$

In[54]:= `a11 = FullSimplify[(1 - k11j) p11[j - 1] - (N - 1) k12j p12[j - 1]]`

Out[54]=
$$\Big(\big(-1 + g\, (1 - j\, (2 + \text{eNom}\, (-2 + N))) + (-1 + \text{eNom})\, g^2\, (-1 + j)\, j\, (1 + \text{eNom}\, (-1 + N))\big)\, \text{p11}[-1 + j] + \text{eNom } g\, (-1 + N)\, \text{p12}[-1 + j]\Big) \Big/ \big((-1 + (-1 + \text{eNom})\, g\, j)\, (1 + g\, j\, (1 + \text{eNom}\, (-1 + N)))\big)$$

In[55]:= `a12 = FullSimplify[(1 - k11j) p12[j - 1] - k12j (p11[j - 1] + (N - 2) p12[j - 1])]`

Out[55]=
$$\text{p12}[-1 + j] + \frac{\text{eNom } g\, \text{p11}[-1 + j] + g\, (1 + g\, j + \text{eNom}\, (-2 + N + g\, j\, (-2 + \text{eNom} + N - \text{eNom } N)))\, \text{p12}[-1 + j]}{(-1 + (-1 + \text{eNom})\, g\, j)\, (1 + g\, j\, (1 + \text{eNom}\, (-1 + N)))}$$

Compute C = A (I - K_j H)^T.

In[56]:= `c00 = FullSimplify[a00 - N k01j a01]`

Out[56]=
$$\text{p00}[-1 + j] + \frac{\text{eNom } g\, N\, (-2\, (1 + g\, j\, (1 + \text{eNom}\, (-1 + N)))\, \text{p01}[-1 + j] + \text{eNom } g\, (\text{p11}[-1 + j] + (-1 + N)\, \text{p12}[-1 + j]))}{(1 + g\, j\, (1 + \text{eNom}\, (-1 + N)))^2}$$

In[57]:= `c01 = FullSimplify[a01 (1 - k11j - (N - 1) k12j)]`

Out[57]=
$$\big((1 + g\, (-1 + j)\, (1 + \text{eNom}\, (-1 + N)))\, ((1 + g\, j\, (1 + \text{eNom}\, (-1 + N)))\, \text{p01}[-1 + j] - \text{eNom } g\, (\text{p11}[-1 + j] + (-1 + N)\, \text{p12}[-1 + j]))\big) \Big/ (1 + g\, j\, (1 + \text{eNom}\, (-1 + N)))^2$$

In[58]:= `c10 = FullSimplify[a10 - k01j (a11 + (N - 1) a12)]`

Out[58]=
$$\Big((1+g(-1+j)(1+\mathrm{eNom}(-1+N)))\big((1+g j(1+\mathrm{eNom}(-1+N)))\,\mathrm{p01}[-1+j]-\mathrm{eNom}\,g\,(\mathrm{p11}[-1+j]+(-1+N)\,\mathrm{p12}[-1+j])\big)\Big)\Big/(1+g j(1+\mathrm{eNom}(-1+N)))^2$$

In[59]:= `c11 = FullSimplify[(1 - k11j) a11 - (N - 1) k12j a12]`

Out[59]=
$$\Bigg(\bigg(1+\frac{g(1-(-1+\mathrm{eNom})g j(1+\mathrm{eNom}(-1+N)))}{(-1+(-1+\mathrm{eNom})g j)(1+g j(1+\mathrm{eNom}(-1+N)))}\bigg)\Big(\big(-1+g(1-j(2+\mathrm{eNom}(-2+N)))+(-1+\mathrm{eNom})g^2(-1+j)j(1+\mathrm{eNom}(-1+N))\big)\mathrm{p11}[-1+j]+\mathrm{eNom}\,g(-1+N)\,\mathrm{p12}[-1+j]\Big)+\mathrm{eNom}\,g(-1+N)\bigg(\mathrm{p12}[-1+j]+\frac{\mathrm{eNom}\,g\,\mathrm{p11}[-1+j]+g(1+g j+\mathrm{eNom}(-2+N+g j(-2+\mathrm{eNom}+N-\mathrm{eNom}\,N)))\,\mathrm{p12}[-1+j]}{(-1+(-1+\mathrm{eNom})g j)(1+g j(1+\mathrm{eNom}(-1+N)))}\bigg)\Bigg)\Bigg/\big((-1+(-1+\mathrm{eNom})g j)(1+g j(1+\mathrm{eNom}(-1+N)))\big)$$

In[60]:= `c12 = FullSimplify[(1 - k11j) a12 - k12j a11 - (N - 2) k12j a12]`

Out[60]=
$$\Big(\mathrm{eNom}\,g\big(-2-g(-1+2j)(2+\mathrm{eNom}(-2+N))+2(-1+\mathrm{eNom})g^2(-1+j)j(1+\mathrm{eNom}(-1+N))\big)\mathrm{p11}[-1+j]+\big(1+2g(-1+2j+\mathrm{eNom}(-1+j)(-2+N))-2(-1+\mathrm{eNom})g^3(-1+j)j(-1+2j+\mathrm{eNom}(-1+j)(-2+N))(1+\mathrm{eNom}(-1+N))+(-1+\mathrm{eNom})^2g^4(-1+j)^2j^2(1+\mathrm{eNom}(-1+N))^2+g^2\big(1+6(-1+j)j+2\,\mathrm{eNom}(-1+j)(-1+3j)(-2+N)+\mathrm{eNom}^2(3+(-3+N)N+j^2(6+(-6+N)N)-2j(5+(-5+N)N))\big)\big)\mathrm{p12}[-1+j]\Big)\Big/\big((-1+(-1+\mathrm{eNom})g j)^2(1+g j(1+\mathrm{eNom}(-1+N)))^2\big)$$

Compute P_j= C + K_j K_j^T. We call the resulting matrix entries *p00Recursive* etc. in contrast to p00[j_] which will represent the closed-form representation of P_j.

In[61]:= `p00Recursive = FullSimplify[c00 + N k01j k01j ]`

Out[61]=
$$\mathrm{p00}[-1+j]+\frac{\mathrm{eNom}\,g\,N\big(-2(1+g j(1+\mathrm{eNom}(-1+N)))\,\mathrm{p01}[-1+j]+\mathrm{eNom}\,g(1+\mathrm{p11}[-1+j]+(-1+N)\,\mathrm{p12}[-1+j])\big)}{(1+g j(1+\mathrm{eNom}(-1+N)))^2}$$

In[62]:= `p01Recursive = FullSimplify[c01 + k01j (k11j + (N - 1) k12j)]`

Out[62]=
$$\Big(\mathrm{eNom}\,g^2(1+\mathrm{eNom}(-1+N))+(1+g(-1+j)(1+\mathrm{eNom}(-1+N)))\big((1+g j(1+\mathrm{eNom}(-1+N)))\,\mathrm{p01}[-1+j]-\mathrm{eNom}\,g(\mathrm{p11}[-1+j]+(-1+N)\,\mathrm{p12}[-1+j])\big)\Big)\Big/(1+g j(1+\mathrm{eNom}(-1+N)))^2$$

In[63]:= `p10Recursive = FullSimplify[c10 + k01j (k11j + (N - 1) k12j)]`

Out[63]=
$$\Big(\mathrm{eNom}\,g^2(1+\mathrm{eNom}(-1+N))+(1+g(-1+j)(1+\mathrm{eNom}(-1+N)))\big((1+g j(1+\mathrm{eNom}(-1+N)))\,\mathrm{p01}[-1+j]-\mathrm{eNom}\,g(\mathrm{p11}[-1+j]+(-1+N)\,\mathrm{p12}[-1+j])\big)\Big)\Big/(1+g j(1+\mathrm{eNom}(-1+N)))^2$$

In[64]:= `p11Recursive = FullSimplify[c11 + k11j k11j + (N - 1) k12j k12j]`

Out[64]=
$$\Bigg(g^2(-1+(-1+\mathrm{eNom})g j(1+\mathrm{eNom}(-1+N)))^2+\mathrm{eNom}^2g^2(-1+N)+(-1+(-1+\mathrm{eNom})g j)(1+g j(1+\mathrm{eNom}(-1+N)))\Bigg(\bigg(1+\frac{g(1-(-1+\mathrm{eNom})g j(1+\mathrm{eNom}(-1+N)))}{(-1+(-1+\mathrm{eNom})g j)(1+g j(1+\mathrm{eNom}(-1+N)))}\bigg)\Big(\big(-1+g(1-j(2+\mathrm{eNom}(-2+N)))+(-1+\mathrm{eNom})g^2(-1+j)j(1+\mathrm{eNom}(-1+N))\big)\mathrm{p11}[-1+j]+\mathrm{eNom}\,g(-1+N)\,\mathrm{p12}[-1+j]\Big)+\mathrm{eNom}\,g(-1+N)\bigg(\mathrm{p12}[-1+j]+\frac{\mathrm{eNom}\,g\,\mathrm{p11}[-1+j]+g(1+g j+\mathrm{eNom}(-2+N+g j(-2+\mathrm{eNom}+N-\mathrm{eNom}\,N)))\,\mathrm{p12}[-1+j]}{(-1+(-1+\mathrm{eNom})g j)(1+g j(1+\mathrm{eNom}(-1+N)))}\bigg)\Bigg)\Bigg)\Bigg/\big((-1+(-1+\mathrm{eNom})g j)^2(1+g j(1+\mathrm{eNom}(-1+N)))^2\big)$$

In[65]:=
```
p12Recursive = FullSimplify[c12 + 2 k11j k12j + (N - 2) k12j k12j]
```

Out[65]=
$$\Big(\mathrm{eNom}\, g^2\, (2 + 2 g j + \mathrm{eNom}\, (-2 + N + 2 g j\, (-2 + \mathrm{eNom} + N - \mathrm{eNom}\, N))) + \mathrm{eNom}\, g \left(-2 - g\, (-1 + 2 j)\, (2 + \mathrm{eNom}\, (-2 + N)) + 2\, (-1 + \mathrm{eNom})\, g^2\, (-1 + j)\, j\, (1 + \mathrm{eNom}\, (-1 + N))\right) \mathrm{p11}[-1 + j] + \left(1 + 2 g\, (-1 + 2 j + \mathrm{eNom}\, (-1 + j)\, (-2 + N)) - 2\, (-1 + \mathrm{eNom})\, g^3\, (-1 + j)\, j\, (-1 + 2 j + \mathrm{eNom}\, (-1 + j)\, (-2 + N))\, (1 + \mathrm{eNom}\, (-1 + N)) + (-1 + \mathrm{eNom})^2\, g^4\, (-1 + j)^2\, j^2\, (1 + \mathrm{eNom}\, (-1 + N))^2 + g^2 \left(1 + 6\, (-1 + j)\, j + 2\, \mathrm{eNom}\, (-1 + j)\, (-1 + 3 j)\, (-2 + N) + \mathrm{eNom}^2 \left(3 + (-3 + N)\, N + j^2\, (6 + (-6 + N)\, N) - 2 j\, (5 + (-5 + N)\, N)\right)\right)\right) \mathrm{p12}[-1 + j]\Big) \Big/ \left((-1 + (-1 + \mathrm{eNom})\, g j)^2\, (1 + g j\, (1 + \mathrm{eNom}\, (-1 + N)))^2\right)$$

(III) Explicit representation of $p_j^{(0,0)}$, $p_j^{(0,1)}$, $p_j^{(1,1)}$, $p_j^{(1,2)}$ (Induction Hypothesis)

Third, we introduce the closed-form representation of P_j (our guess) which we will prove by induction given the recursive equation above.

In[66]:=
```
p00[j_] = FullSimplify[
  (g ((d + eNom) (1 + g (j - eNom j))^2 + eNom^2 g j (1 - (-1 + d + eNom) g j) N)) / (1 + g j (1 + eNom (-1 + N)))^2]
```

Out[66]=
$$\frac{g \left((d + \mathrm{eNom})\, (1 + g\, (j - \mathrm{eNom}\, j))^2 + \mathrm{eNom}^2\, g j\, (1 - (-1 + d + \mathrm{eNom})\, g j)\, N\right)}{(1 + g j\, (1 + \mathrm{eNom}\, (-1 + N)))^2}$$

In[67]:=
```
p01[j_] = FullSimplify[(g (d + eNom + d g j + eNom g j (1 + eNom (-1 + N)))) / (1 + g j (1 + eNom (-1 + N)))^2]
```

Out[67]=
$$\frac{g\, (d + \mathrm{eNom} + d g j + \mathrm{eNom}\, g j\, (1 + \mathrm{eNom}\, (-1 + N)))}{(1 + g j\, (1 + \mathrm{eNom}\, (-1 + N)))^2}$$

In[68]:=
```
p10[j_] = FullSimplify[(g (d + eNom + d g j + eNom g j (1 + eNom (-1 + N)))) / (1 + g j (1 + eNom (-1 + N)))^2]
```

Out[68]=
$$\frac{g\, (d + \mathrm{eNom} + d g j + \mathrm{eNom}\, g j\, (1 + \mathrm{eNom}\, (-1 + N)))}{(1 + g j\, (1 + \mathrm{eNom}\, (-1 + N)))^2}$$

In[69]:=
```
check = p01[j] - p01[j]
```

Out[69]= 0

In[70]:=
```
p11[j_] = FullSimplify[
  (g ((-1 + (-1 + eNom) g j) (1 + g j (1 + eNom (-1 + N))) (-1 + (-1 + eNom) g j (1 + eNom (-1 + N))) - d eNom g j
      (2 + g j (2 + eNom (-2 + N))) (-1 + N))) / ((-1 + (-1 + eNom) g j)^2 (1 + g j (1 + eNom (-1 + N)))^2)]
```

Out[70]=
$$\left(g\, \left((-1 + (-1 + \mathrm{eNom})\, g j)\, (1 + g j\, (1 + \mathrm{eNom}\, (-1 + N)))\, (-1 + (-1 + \mathrm{eNom})\, g j\, (1 + \mathrm{eNom}\, (-1 + N))) - d\, \mathrm{eNom}\, g j\, (2 + g j\, (2 + \mathrm{eNom}\, (-2 + N)))\, (-1 + N)\right)\right) \Big/ \left((-1 + (-1 + \mathrm{eNom})\, g j)^2\, (1 + g j\, (1 + \mathrm{eNom}\, (-1 + N)))^2\right)$$

In[71]:=
```
p12[j_] = FullSimplify[
  (g (d (1 + g j (2 + g j (1 + eNom^2 (-1 + N)))) - eNom (-1 + (-1 + eNom) g j) (1 + g j (1 + eNom (-1 + N))))) /
   ((-1 + (-1 + eNom) g j)^2 (1 + g j (1 + eNom (-1 + N)))^2)]
```

Out[71]=
$$\frac{g \left(d \left(1 + g j \left(2 + g j \left(1 + \mathrm{eNom}^2\, (-1 + N)\right)\right)\right) - \mathrm{eNom}\, (-1 + (-1 + \mathrm{eNom})\, g j)\, (1 + g j\, (1 + \mathrm{eNom}\, (-1 + N)))\right)}{(-1 + (-1 + \mathrm{eNom})\, g j)^2\, (1 + g j\, (1 + \mathrm{eNom}\, (-1 + N)))^2}$$

(IV) Explicit representation of P_j is true for j=0. (Base Case)

For j=1, the closed-form representation results in the true initial noise variance given by eq. (9) (cf. also eq. (20)). Recall that $\epsilon = \bar{\epsilon} + \delta$ which represents the real distrubance partitioning.

In[72]:=
```
p00[0]
```

Out[72]= $(d + \mathrm{eNom})\, g$

In[73]:=
```
p10[0]
```

Out[73]= $(d + \mathrm{eNom})\, g$

In[74]:=
```
p11[0]
```

Out[74]= g

In[75]:=
```
p12[0]
```
Out[75]=
```
(d + eNom) g
```

(V) Explicit representation of P_j satisfies the recursive equations. (Inductive Step)

We plug in p00[j-1], p01[j-1], p10[j-1], p11[j-1], p12[j-1] into the derived recursive equations *p00Recursive, p01Recursive* etc. and show that the value equals p00[j], p01[j], p10[j], p11[j], p12[j]. q.e.d

In[76]:=
```
FullSimplify[p00Recursive - p00[j]]
```
Out[76]=
```
0
```
In[77]:=
```
FullSimplify[p01Recursive - p01[j]]
```
Out[77]=
```
0
```
In[78]:=
```
FullSimplify[p10Recursive - p10[j]]
```
Out[78]=
```
0
```
In[79]:=
```
FullSimplify[p11Recursive - p11[j]]
```
Out[79]=
```
0
```
In[80]:=
```
FullSimplify[p12Recursive - p12[j]]
```
Out[80]=
```
0
```