Supplementary Material on Quadrotor Application: Model Predictive Path-Following for Constrained Differentially Flat Systems

Melissa Greeff and Angela P. Schoellig

***Abstract*— Supplementary material detailing application of presented methodology *Flatness Approach to Predictive Path-Following (FAPP)* - submitted by M.Greeff and A.P. Schoellig to ICRA2018 - to a quadrotor.**

I. INTRODUCTION

This supplementary material details why and how a presented methodology *Flatness Approach to Predictive Path-Following (FAPP)* can be applied to a quadrotor. We briefly describe the physical derivation of the nonlinear quadrotor model. We then show that this nonlinear model is indeed differentially flat and as such we can apply feedforward linearization. We similarly show the associated path-attached virtual quadrotor model.

II. NONLINEAR MODEL

We can derive the *standard quadrotor model*, $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$, [1], [2] using newtonian mechanics by consider the rigid-body dynamics of a quadrotor. We make the following assumptions in the derivation:

- Aerodynamic effects, including drag, are ignored.
- The center of mass of the quadrotor is aligned with the origin of the body axis.
- The quadrotor has a symmetric structure.
- Gyroscopic forces are negligeable.

We define geometric variables in Fig. 1. The origin of the body frame, F_B, is placed at the centre of mass of the quadrotor. The body frame is aligned such that X_B and Y_B are the physical axes of the quadrotor and Z_B is normal to the hover plane. The linear position of the origin of the body frame with respect to the inertial frame, F_I is defined as $\xi = (x, y, z)$. We define the orientation of the body frame with respect to the inertial frame, $\mathbf{R} = Z_\psi Y_\theta X_\phi$ where the euler angles $\eta = (\phi, \theta, \psi)$ are the roll, pitch and yaw angles respectively.

Positional Dynamics: We define the thrust in the body frame: $\mathbf{T}^B := (0, 0, T)$. Considering then that $\mathbf{T}^I = \mathbf{R}\mathbf{T}^B$, where $\mathbf{T}^I$ is the thrust in the inertial frame, newton's second law gives:

$$m\ddot{\xi} = m\mathbf{G} + \mathbf{R}_B^I \mathbf{T}^B, \tag{1}$$

The authors are with the Dynamic Systems Lab (www.dynsyslab.org) at the University of Toronto Institute for Aerospace Studies (UTIAS), Canada. melissa.greeff@mail.utoronto.ca, schoellig@utias.utoronto.ca

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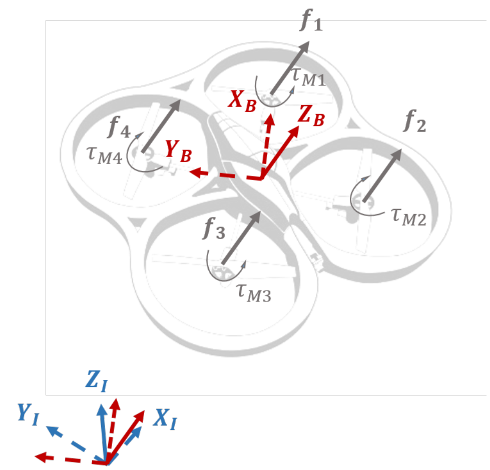

Fig. 1. Quadrotor Frames for Position and Rotational Dynamics

where $\mathbf{G} = [0, 0, g = 9.8] \quad m/s^2$ is the gravity vector, $\ddot{\xi}$ is the translational acceleration and m is the mass of the quadrotor.

Rotational Dynamics: We define the angular velocities in the body frame as $\boldsymbol{\upsilon} = (p, q, r)$. Then $\boldsymbol{\upsilon} = \mathbf{J}_\eta \dot{\boldsymbol{\eta}}$ where $\mathbf{J}_\eta$ is the angular velocity Jacobian matrix:

$$\mathbf{J}_\eta = \begin{bmatrix} 1 & 0 & -S_\theta \\ 0 & C_\phi & C_\theta S_\phi \\ 0 & -S_\phi & C_\theta C_\phi \end{bmatrix}$$

Given that the body frame is aligned with the physical axes of the rotor, we assume the inertial matrix $\mathbf{I}$ (with respect to the body frame, F_B) is diagonal:

$$\mathbf{I} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

We define the torques about the respective body frame axes as $\boldsymbol{\tau}^B := (\tau_\phi, \tau_\theta, \tau_\psi)$. Again, considering balancing torques in the body frame, from newton's second law we have the angular dynamics:

$$\mathbf{I}\dot{\boldsymbol{\upsilon}} = \boldsymbol{\tau}^B - \boldsymbol{\upsilon} \times (\mathbf{I}\boldsymbol{\upsilon}) \tag{2}$$

where $\boldsymbol{\upsilon} \times (\mathbf{I}\boldsymbol{\upsilon})$ are the centripetal forces.

Defining the state $\mathbf{x} = (x, y, z, \dot{x}, \dot{y}, \dot{z}, \mathbf{R}, p, q, r)$ and input $\mathbf{u} = (u_1, u_2, u_3, u_4) = (T, \tau_\phi, \tau_\theta, \tau_\psi)$, using (1) and (2), we can write the quadrotor dynamics in the form $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$ where:

$$f(\mathbf{x},\mathbf{u}) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ 0 \\ 0 \\ -g \\ \mathbf{R}\times[p,q,r]^T \\ (I_y-I_z)qr/I_x \\ (I_z-I_x)pr/I_y \\ (I_x-I_y)pq/I_z \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{m}\mathbf{R}_{1,3} & 0 & 0 & 0 \\ \frac{1}{m}\mathbf{R}_{2,3} & 0 & 0 & 0 \\ \frac{1}{m}\mathbf{R}_{3,3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{I_x} & 0 & 0 \\ 0 & 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & 0 & \frac{1}{I_z} \end{bmatrix}\mathbf{u} \tag{3}$$

We also include a model of the dynamics of an inner loop controller which takes inputs $\mathbf{u}_{cmd} = (\dot{z}_{cmd}, \phi_{cmd}, \theta_{cmd}, r_{cmd})$, where $\dot{z}_{cmd}$ is a commanded velocity in z, ϕ_{cmd} and θ_{cmd} are commanded roll and pitch angles and r_{cmd} is a commanded yaw velocity in the body frame, and outputs $\mathbf{u}$:

$$\ddot{z}_d = \frac{1}{\tau_{Iz}}(\dot{z}_{cmd} - \dot{z}) \tag{4a}$$

$$u_1 = m\frac{g+\ddot{z}_d}{\mathbf{R}_{3,3}} \tag{4b}$$

$$\psi_d = \psi + \tau_{I\psi} r_{cmd} \tag{4c}$$

$$r_d = r_{cmd} \tag{4d}$$

$$p_d = \frac{\mathbf{R}_{2,1}(\mathbf{R}_{1,3,d} - \mathbf{R}_{1,3}) - \mathbf{R}_{1,1}(\mathbf{R}_{2,3,d} - \mathbf{R}_{2,3})}{\mathbf{R}_{3,3}\cdot\tau_{rp}} \tag{4e}$$

$$q_d = \frac{\mathbf{R}_{2,2}(\mathbf{R}_{1,3,d} - \mathbf{R}_{1,3}) - \mathbf{R}_{1,2}(\mathbf{R}_{2,3,d} - \mathbf{R}_{2,3})}{\mathbf{R}_{3,3}\cdot\tau_{rp}} \tag{4f}$$

$$\mathbf{R}_{1,3,d} = s_{\psi_d}s_{\phi_{cmd}} + c_{\psi_d}c_{\phi_{cmd}}s_{\theta_{cmd}} \tag{4g}$$

$$\mathbf{R}_{2,3,d} = s_{\psi_d}c_{\phi_{cmd}}s_{\theta_{cmd}} - c_{\psi_d}s_{\phi_{cmd}} \tag{4h}$$

$$u_2 = \frac{p_d - p}{\tau_p} \tag{4i}$$

$$u_3 = \frac{q_d - q}{\tau_q} \tag{4j}$$

$$u_4 = \frac{r_d - r}{\tau_r} \tag{4k}$$

where $\tau_{Iz}, \tau_{I\psi}, \tau_{rp}, \tau_r, \tau_p, \tau_q$ are identified inner loop control parameters [1].

III. FEEDFORWARD LINEARIZATION

A. Demonstration of Differential Flatness

Recall the formal definition of differential flatness.

Definition 1. A nonlinear system model (1) is *differentially flat* if there exists $\boldsymbol{\zeta}(t) \in \mathbb{R}^m$, whose components are differentially independent, such that the following holds [3]:

$$\boldsymbol{\zeta} = \Lambda(\mathbf{x}, \mathbf{u}, \dot{\mathbf{u}}, \ldots, \mathbf{u}^{(\delta)}) \tag{5}$$

$$\mathbf{x} = \Phi(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}, \ldots, \boldsymbol{\zeta}^{(\rho-1)}) \tag{6}$$

[1] $s_\alpha = \sin(\alpha)$ and $c_\alpha = \cos(\alpha)$

$$\mathbf{u} = \Psi^{-1}(\boldsymbol{\zeta}, \dot{\boldsymbol{\zeta}}, \ldots, \boldsymbol{\zeta}^{(\rho)}) \tag{7}$$

where Λ, Φ and Ψ^{-1} are smooth functions, δ and ρ are the maximum orders of the derivatives of $\mathbf{u}$ and $\boldsymbol{\zeta}$ needed to describe the system and $\boldsymbol{\zeta} = [\zeta_1, \ldots, \zeta_m]^T$ is called the flat output.

The differential flatness of the quadrotor model (3) is demonstrated in [1] for flat outputs $\boldsymbol{\zeta} = (x, y, z, \psi)$. We now show that the quadrotor with inner loop control dynamics does not change the differential flatness property of the original nonlinear quadrotor model. In order to show this we need to show conditions (5)-(7) for the combined quadrotor and inner loop dynamic model. We consider the same flat outputs $\boldsymbol{\zeta} = (x, y, z, \psi)$ which are differentially independent, i.e., any output cannot be written in terms of the derivatives of the other outputs:

Condition (5): Notice that given that the flat outputs $\boldsymbol{\zeta}$ are comprised of some terms of state $\mathbf{x}$, condition (5) is shown by definition.

Condition (6): Similarly, condition (6) holds for the translational states, i.e., (x, y, z) and $(\dot{x}, \dot{y}, \dot{z})$, of $\mathbf{x}$ by definition of the flat outputs. We are then left to derive $\mathbf{R}$ and (p, q, r)in terms of the flat state. We begin by writing $\mathbf{R}$ in terms of its column vectors and considering the translational acceleration (from (1)) in the standard quadrotor model:

$$\mathbf{a} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z}+g \end{bmatrix} = \begin{bmatrix} \ddot{\zeta}_1 \\ \ddot{\zeta}_2 \\ \ddot{\zeta}_3+g \end{bmatrix} = \frac{T}{m}\mathbf{z}_\mathbf{B}^\mathbf{I} \tag{8}$$

where we defined $\mathbf{R} \equiv \mathbf{R}_\mathbf{B}^\mathbf{I} = [\mathbf{x}_\mathbf{B}^\mathbf{I} \quad \mathbf{y}_\mathbf{B}^\mathbf{I} \quad \mathbf{z}_\mathbf{B}^\mathbf{I}]$ in terms of column vectors. Using the fact that $\mathbf{z}_\mathbf{B}^\mathbf{I}$ is a unit vector and $\frac{T}{m}$ is a scalar we have $\mathbf{z}_\mathbf{B}^\mathbf{I} = \frac{\mathbf{a}}{||\mathbf{a}||}$. As in [1], we introduce an intermediate frame C produced by rotation, $\mathbf{R}_\mathbf{C}^\mathbf{I} = [\mathbf{x}_\mathbf{C}^\mathbf{I} \quad \mathbf{y}_\mathbf{C}^\mathbf{I} \quad \mathbf{z}_\mathbf{C}^\mathbf{I}]$, of ψ about the inertial z-axis and make the observation that $\mathbf{x}_\mathbf{C}^\mathbf{I}$ lies in the $\mathbf{z}_\mathbf{B}^\mathbf{I} - \mathbf{x}_\mathbf{B}^\mathbf{I}$ plane. Given that $\mathbf{y}_\mathbf{B}^\mathbf{I}$ is normal to the $\mathbf{z}_\mathbf{B}^\mathbf{I} - \mathbf{x}_\mathbf{B}^\mathbf{I}$ plane we can determine $\mathbf{y}_\mathbf{B}^\mathbf{I} = \frac{\mathbf{z}_\mathbf{B}^\mathbf{I} \times \mathbf{x}_\mathbf{C}^\mathbf{I}}{||\mathbf{z}_\mathbf{B}^\mathbf{I} \times \mathbf{x}_\mathbf{C}^\mathbf{I}||}$ and $\mathbf{x}_\mathbf{B}^\mathbf{I} = \mathbf{y}_\mathbf{B}^\mathbf{I} \times \mathbf{z}_\mathbf{B}^\mathbf{I}$. Consequently, we have shown $\mathbf{R}$ to be a function of the flat outputs and some of their derivatives.

We now show the same for (p, q, r) by differentiating (8):

$$\dot{\mathbf{a}} = \frac{\dot{T}}{m}\mathbf{z}_\mathbf{B}^\mathbf{I} + \frac{T}{m}(\boldsymbol{\omega}_\mathbf{B}^\mathbf{I} \times \mathbf{z}_\mathbf{B}^\mathbf{I}), \tag{9}$$

where $\boldsymbol{\omega}_\mathbf{B}^\mathbf{I} = p\mathbf{x}_\mathbf{B}^\mathbf{I} + q\mathbf{y}_\mathbf{B}^\mathbf{I} + r\mathbf{z}_\mathbf{B}^\mathbf{I}$. Plugging the defintion of $\boldsymbol{\omega}_\mathbf{B}^\mathbf{I}$ into (9) reduces:

$$\dot{\mathbf{a}} = \frac{\dot{T}}{m}\mathbf{z}_\mathbf{B}^\mathbf{I} + \frac{T}{m}(-p\mathbf{y}_\mathbf{B}^\mathbf{I} + q\mathbf{x}_\mathbf{B}^\mathbf{I}), \tag{10}$$

From the third component in (8) we have $\frac{T}{m} = \frac{\ddot{z}+g}{\mathbf{R}_{3,3}}$ which after differentiation gives:

$$\frac{\dot{T}}{m} = \frac{\dddot{z}\mathbf{R}_{3,3} - (\ddot{z}+g)\dot{\mathbf{R}}_{3,3}}{\mathbf{R}_{3,3}^\mathbf{2}} \tag{11}$$

where from the standard quadrotor model $\dot{\mathbf{R}} = \mathbf{R} \times [p, q, r]^T$ and therefore $\dot{\mathbf{R}}_{3,3} = \mathbf{R}_{3,1} q - \mathbf{R}_{3,2} p$. Plugging $\frac{T}{m}$ and $\frac{\dot{T}}{m}$ from (11) into (10) allows us to solve for p and q as $[p, q]^T = \mathbf{M_1}^{-1} \mathbf{n_1}$ where:

$$\mathbf{M_1} = \begin{bmatrix} \frac{\mathbf{R}_{3,2}\mathbf{R}_{1,3} - \mathbf{R}_{1,2}\mathbf{R}_{3,3}}{\mathbf{R}_{3,3}} & \frac{\mathbf{R}_{1,1}\mathbf{R}_{3,3} - \mathbf{R}_{3,1}\mathbf{R}_{1,3}}{\mathbf{R}_{3,3}} \\ \frac{\mathbf{R}_{3,2}\mathbf{R}_{2,3} - \mathbf{R}_{2,2}\mathbf{R}_{3,3}}{\mathbf{R}_{3,3}} & \frac{\mathbf{R}_{2,1}\mathbf{R}_{3,3} - \mathbf{R}_{3,1}\mathbf{R}_{2,3}}{\mathbf{R}_{3,3}} \end{bmatrix}$$

$$\mathbf{n_1} = \begin{bmatrix} \frac{\dddot{x}\mathbf{R}_{3,3} - \dddot{z}\mathbf{R}_{1,3}}{\ddot{z}+g} \\ \frac{\dddot{y}\mathbf{R}_{3,3} - \dddot{z}\mathbf{R}_{2,3}}{\ddot{z}+g} \end{bmatrix}$$

Furthermore, as in [**?**], considering the third component of $\boldsymbol{\omega}_\mathbf{B}^\mathbf{I} = \boldsymbol{\omega}_\mathbf{B}^\mathbf{C} + \boldsymbol{\omega}_\mathbf{C}^\mathbf{I}$ gives $r = \dot{\psi}\mathbf{R}_{3,3}$. We have shown condition (6) of differential flatness.

Condition (7): We demonstrate condition (7) by showing $\mathbf{u}_{cmd}$ as a function of the flat outputs $\boldsymbol{\zeta}$ and their derivatives. First, we substitute u_1 in (4b) into (8) and observe that $\ddot{z}_d = \ddot{z}$. We substitute this result into (4a) which after rearranging gives:

$$\dot{z}_{cmd} = \dot{z} + \frac{1}{\tau_{Iz}} \ddot{z} \tag{12}$$

which is a function then of the flat state z up to its second derivative.

Next we find ϕ_{cmd} and θ_{cmd} by first taking the derivative of (10) with respect to time:

$$\ddot{\mathbf{a}} = \frac{\ddot{T}}{m}\mathbf{z}_\mathbf{B}^\mathbf{I} + 2\frac{\dot{T}}{m}(\boldsymbol{\omega}_\mathbf{B}^\mathbf{I} \times \mathbf{z}_\mathbf{B}^\mathbf{I}) + \frac{T}{m}\boldsymbol{\omega}_\mathbf{B}^\mathbf{I} \times \boldsymbol{\omega}_\mathbf{B}^\mathbf{I} \times \mathbf{z}_\mathbf{B}^\mathbf{I} + \frac{T}{m}\boldsymbol{\alpha}_\mathbf{B}^\mathbf{I} \times \mathbf{z}_\mathbf{B}^\mathbf{I}, \tag{13}$$

where $\boldsymbol{\alpha}_\mathbf{B}^\mathbf{I} = \dot{p}\mathbf{x}_\mathbf{B}^\mathbf{I} + \dot{q}\mathbf{y}_\mathbf{B}^\mathbf{I} + \dot{r}\mathbf{z}_\mathbf{B}^\mathbf{I}$. We notice that we can already compute the terms $2\dot{T}(\boldsymbol{\omega}_\mathbf{B}^\mathbf{I} \times \mathbf{z}_\mathbf{B}^\mathbf{I})$ and $T\boldsymbol{\omega}_\mathbf{B}^\mathbf{I} \times \boldsymbol{\omega}_\mathbf{B}^\mathbf{I} \times \mathbf{z}_\mathbf{B}^\mathbf{I}$ in terms of flat outputs and their derivatives. We also have that $\boldsymbol{\alpha}_\mathbf{B}^\mathbf{I} \times \mathbf{z}_\mathbf{B}^\mathbf{I} = -\dot{p}\mathbf{y}_\mathbf{B}^\mathbf{I} + \dot{q}\mathbf{x}_\mathbf{B}^\mathbf{I}$ from the definition of $\boldsymbol{\alpha}_\mathbf{B}^\mathbf{I}$. By taking the derivative of (11) we can compute $\ddot{T}$, which after incorporating $\dot{\mathbf{R}}_{3,1} = \mathbf{R}_{1,1}q - \mathbf{R}_{1,2}p$ and $\dot{\mathbf{R}}_{3,2} = \mathbf{R}_{2,1}q - \mathbf{R}_{2,2}p$, we can plug into our relation for $\ddot{\mathbf{a}}$ in (13) along with the remaining known quantities to solve for $\dot{p}$ and $\dot{q}$ in terms of the flat outputs and their derivatives. Taking the derivative of $\dot{\psi}\mathbf{R}_{3,3}$ gives the relation for $\dot{r}$.

Now considering the rotational dynamics (2) in the standard quadrotor model and using our relations for p, q, r and $\dot{p}, \dot{q}, \dot{r}$ we can determine u_2, u_3, u_4 in terms of the flat outputs and their derivatives. Plugging u_2, u_3, u_4 into (4i)-(4k) we can solve for p_d, q_d, r_d in terms of the flat outputs and their derivatives. From (4d) we have $\dot{\psi}_{cmd} = r_d$. We use (4g) and (4h) to find $\mathbf{R}_{1,3,d}$ and $\mathbf{R}_{2,3,d}$ which when using (4c) in (4e) and (4f) allows us to solve for θ_{cmd} and ϕ_{cmd} in terms of the flat outputs and their derivatives. We have demonstrated that conditions (5)-(7) hold and the standard quadrotor including inner loop controller model is differentially flat.

B. Feedforward Linearization

From the demonstration of the differential flatness of the quadrotor and inner loop dynamics, we can define flat state:

$$\mathbf{z} = (x, \dot{x}, \ddot{x}, \dddot{x}, y, \dot{y}, \ddot{y}, \dddot{y}, z, \dot{z}, \ddot{z}, \dddot{z}, \psi, \dot{\psi})$$

and flat input

$$\mathbf{v} = (x^{(4)}, y^{(4)}, z^{(4)}, \ddot{\psi}).$$

As explained in [], we can then equivalently rewrite the nonlinear quadrotor model (3) as:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{v} \tag{14}$$

$$\mathbf{v} = \Psi(\mathbf{z}, \mathbf{u}, \dot{\mathbf{u}}, \ldots, \mathbf{u}^{(\sigma)}), \tag{15}$$

where (14) is the linear flat model and the inverse of (15) is given by (7), which we can determine as shown in the preceding section by demonstrating condition (7) for the nonlinear quadrotor model.

We therefore consider (14) in the model predictive control framework which outputs the next desired flat state $\mathbf{z}_d$ and input $\mathbf{v}_d$ which we then feed through the inverse (7) as:

$$\mathbf{u}_{cmd} = \Psi^{-1}(\mathbf{z}_d, \mathbf{v}_d).$$

IV. PATH-FOLLOWING

We consider a path-attached virtual quadrotor, with an associated path dynamic model the path-attached virtual quadrotor state $\mathbf{s}(t) \in \mathbb{R}^\rho$, the path-attached virtual quadrotor input $w(t) \in \mathbb{R}$ and g a smooth function, attached to a parametrized geometric path in the flat output space:

$$P = \{\boldsymbol{\zeta}_{ref} \in \mathbb{R}^m | \quad \boldsymbol{\zeta}_{ref} = p(\theta(t)), \quad \theta \in [\theta_0, \theta_1]\}. \tag{16}$$

Considering $\rho = 4$ (the maximum derivative of flat output in derivation of condition (7)), the associated path dynamic model is given by the linear path dynamic model for a virtual quadrotor vehicle is given by $\dot{\mathbf{s}} = \mathbf{A_p}\mathbf{s} + \mathbf{B_p}w$ [4] where:

$$\mathbf{A_p} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B_p} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

with $\mathbf{s} = (\theta, \dot{\theta}, \ddot{\theta}, \dddot{\theta})$.

V. PREDICTIVE CONTROL

We recall the aim of FAPP: Given a quadrotor that can be represented by a *differentially flat* nonlinear model (3) and a geometric path to be followed (16), design the flat input $\mathbf{v}(t)$ and path-attached virtual quadrotor input $w(t)$ such that the following is satisfied: At every time step k, solve the OCP:

$$\min_{\hat{\mathbf{v}}, \hat{\mathbf{w}}} J(\hat{\mathbf{z}}, \hat{\mathbf{v}}, \hat{\mathbf{s}}, \hat{\mathbf{w}}), \tag{17}$$

where we consider the sequence of predicted flat states $\hat{\mathbf{z}}$, flat inputs $\hat{\mathbf{v}}$, path states $\hat{\mathbf{s}}$ and path inputs $\hat{\mathbf{w}}$. This OCP is subject to the equivalent *linear flat model* in (14), the *linear path dynamic model* in (which gives a corresponding reference flat state, $\mathbf{z}_{ref}$, through path parameterization (14)) and linear constraints on the optimization variables (the flat inputs and path inputs).

Straight Line Path Following: We consider the aim of FAPP and optimization problem in equation (17). Then for a straight line Bézier curves, the reference flat state $\boldsymbol{\zeta}_{ref} = p(\theta) = \sum_{j=0}^{q} \binom{q}{j} (1-\theta)^{q-j}\theta^{j}\mathbf{P_j}, \quad \theta \in [0,1]$, where $q = 1$:

$$\boldsymbol{\zeta}_{ref} = \begin{bmatrix} x_{ref}(\theta) \\ y_{ref}(\theta) \\ z_{ref}(\theta) \\ \psi_{ref}(\theta) \end{bmatrix} = \mathbf{P_0}(1-\theta) + \mathbf{P_1}(\theta). \tag{18}$$

Taking up to the third derivative of (18), we can determine the reference flat state:

$$\mathbf{z}_{ref} = \mathbf{\Pi s} + \mathbf{\Pi_0}$$

where:

$$\mathbf{\Pi} = \begin{bmatrix} P_{1,x}-P_{0,x} & 0 & 0 & 0 \\ 0 & P_{1,x}-P_{0,x} & 0 & 0 \\ 0 & 0 & P_{1,x}-P_{0,x} & 0 \\ 0 & 0 & 0 & P_{1,x}-P_{0,x} \\ P_{1,y}-P_{0,y} & 0 & 0 & 0 \\ 0 & P_{1,y}-P_{0,y} & 0 & 0 \\ 0 & 0 & P_{1,y}-P_{0,y} & 0 \\ 0 & 0 & 0 & P_{1,y}-P_{0,y} \\ P_{1,z}-P_{0,z} & 0 & 0 & 0 \\ 0 & P_{1,z}-P_{0,z} & 0 & 0 \\ 0 & 0 & P_{1,z}-P_{0,z} & 0 \\ 0 & 0 & 0 & P_{1,z}-P_{0,z} \\ P_{1,\psi}-P_{0,\psi} & 0 & 0 & 0 \\ 0 & P_{1,\psi}-P_{0,\psi} & 0 & 0 \end{bmatrix}$$

$$\mathbf{\Pi_0} = \begin{bmatrix} P_{0,x} \\ 0 \\ 0 \\ 0 \\ P_{0,y} \\ 0 \\ 0 \\ 0 \\ P_{0,z} \\ 0 \\ 0 \\ 0 \\ P_{0,\psi} \\ 0 \end{bmatrix}$$

where $\mathbf{P_0} = (P_{0,x}, P_{0,y}, P_{0,z}, P_{0,psi})$ and $\mathbf{P_1} = (P_{1,x}, P_{1,y}, P_{1,z}, P_{1,psi})$ are the control points for a straight line Bézier curve and we vary the path parameter θ (based on distance along the path) between 0 and 1.

We then discretize the linear flat model (14) and linear path model:

$$\mathbf{z}_{k+1} = \mathbf{A_d}\mathbf{z}_k + \mathbf{B_d}\mathbf{v}_k.$$

$$\mathbf{s}_{k+1} = \mathbf{A_{pd}}\mathbf{s}_k + \mathbf{B_{pd}}w_k.$$

Given a current measured flat state, $\mathbf{z}_0$, and a current path state, $\mathbf{s}_0$, we write lifted forms, for N prediction steps, of our discretized models:

$$\underbrace{\begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \\ \mathbf{z}_N \end{bmatrix}}_{\hat{\mathbf{z}}} = \underbrace{\begin{bmatrix} \mathbf{A_d} \\ \mathbf{A_d^2} \\ \vdots \\ \mathbf{A_d^N} \end{bmatrix}}_{\hat{\mathbf{A}}} \mathbf{z}_0 + \underbrace{\begin{bmatrix} \mathbf{B_d} & 0 & 0\dots0 \\ \mathbf{A_d B_d B_d} & & 0\dots0 \\ \vdots & \ddots & \ddots \\ \mathbf{A_d^{N-1}B_d} & \dots & \mathbf{A_d B_d B_d} \end{bmatrix}}_{\hat{\mathbf{B}}} \underbrace{\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_N \end{bmatrix}}_{\hat{\mathbf{v}}}, \tag{19}$$

$$\hat{\mathbf{s}} = \hat{\mathbf{A}}_\mathbf{p}\mathbf{s}_0 + \hat{\mathbf{B}}_\mathbf{p}\hat{\mathbf{w}}. \tag{20}$$

A potential form of the cost function $\mathbf{J}$ in (17) is:

$$\mathbf{J} = \frac{1}{2}\sum_{i=1}^{N}(\mathbf{z} - (\mathbf{\Pi s} + \mathbf{\Pi_0}))^T\mathbf{Q}(\mathbf{z} - (\mathbf{\Pi s} + \mathbf{\Pi_0}))$$
$$+\frac{1}{2}\sum_{i=1}^{N}(\mathbf{z} - \mathbf{z}_{cmd})^T\mathbf{S}(\mathbf{z} - \mathbf{z}_{cmd})$$
$$+\frac{1}{2}\sum_{i=1}^{N}\mathbf{v_i}^T\mathbf{R}\mathbf{v_i} + \frac{1}{2}\sum_{i=1}^{N}w_i{}^T R_p w_i$$

where the first term weights the positional error between the quadrotor and the path-attached virtual quadrotor, the second term more the quadrotor forward by weighting the velocity error between the quadrotor velocity and a desired velocity. The final two terms are regularization terms on the snaps (flat inputs) of the quadrotor and virtual quadrotor.

We then define the expanded forms of our weight matrices: $\hat{\mathbf{Q}} \in \mathbb{R}^{N\bar{\rho}\times N\bar{\rho}}$ where $\hat{\mathbf{Q}} = diag(\mathbf{Q})$ and similarly for $\hat{\mathbf{S}}$, $\hat{\mathbf{R}}$ and $\hat{\mathbf{R}}_\mathbf{p}$. Further, we define: $\hat{\mathbf{\Pi}} = diag(\mathbf{\Pi})$, $\hat{\mathbf{\Pi}}_\mathbf{0} = [\mathbf{\Pi_0}, \mathbf{\Pi_0}, \dots, \mathbf{\Pi_0}]^T$ and $\hat{\mathbf{z}}_{cmd} = [\mathbf{z}_{cmd,1}, \mathbf{z}_{cmd,2}, \dots, \mathbf{z}_{cmd,N}]^T$. We rewrite (17) using (19) and (20) and these expanded matrices. Expanding we obtain the quadratic cost:

$$J(\tilde{\mathbf{v}}) = \frac{1}{2}\tilde{\mathbf{v}}^T\mathbf{H}\tilde{\mathbf{v}} + \mathbf{f}^T\tilde{\mathbf{v}}$$

where $\tilde{\mathbf{v}} = \begin{bmatrix} \hat{\mathbf{v}}^T & \hat{\mathbf{w}}^T \end{bmatrix}^T$ with

$$\mathbf{H} = \begin{bmatrix} \hat{\mathbf{B}}^T\hat{\mathbf{Q}}\hat{\mathbf{B}} + \hat{\mathbf{B}}^T\hat{\mathbf{S}}\hat{\mathbf{B}} + \hat{\mathbf{R}} & -\hat{\mathbf{B}}^T\hat{\mathbf{Q}}\hat{\mathbf{\Pi}}\hat{\mathbf{B}}_\mathbf{p} \\ -\hat{\mathbf{B}}_\mathbf{p}^T\hat{\mathbf{\Pi}}^T\hat{\mathbf{Q}}\hat{\mathbf{B}} & \hat{\mathbf{B}}_\mathbf{p}^T(\hat{\mathbf{\Pi}}^T\hat{\mathbf{Q}}\hat{\mathbf{\Pi}})\hat{\mathbf{B}}_\mathbf{p} + \hat{\mathbf{R}}_\mathbf{p} \end{bmatrix},$$

$$\mathbf{f} = \begin{bmatrix} \mathbf{f_1} \\ \mathbf{f_2} \end{bmatrix},$$

with $\mathbf{f_1} = (\mathbf{z_0}{}^T\hat{\mathbf{A}}^T - \mathbf{s_0}{}^T\hat{\mathbf{A}}_\mathbf{p}^T\hat{\mathbf{\Pi}}^T - \hat{\mathbf{\Pi}}_\mathbf{0}^T)\hat{\mathbf{Q}}\hat{\mathbf{B}} + (\mathbf{z_0}{}^T\hat{\mathbf{A}}^T - \mathbf{z^{cmd}}^T)\hat{\mathbf{S}}\hat{\mathbf{B}}$ and $\mathbf{f_2} = -\mathbf{z_0}{}^T\hat{\mathbf{A}}^T\hat{\mathbf{Q}}\hat{\mathbf{\Pi}}\hat{\mathbf{B}}_\mathbf{p} + \mathbf{s_0}{}^T\hat{\mathbf{A}}_\mathbf{p}^T\hat{\mathbf{\Pi}}^T\hat{\mathbf{Q}}\hat{\mathbf{\Pi}}\hat{\mathbf{B}}_\mathbf{p} + \hat{\mathbf{\Pi}}_\mathbf{0}^T\hat{\mathbf{Q}}\hat{\mathbf{\Pi}}\hat{\mathbf{B}}_\mathbf{p}$.

VI. CONSTRAINTS

We generally considered two sets of constraints on the quadrotor: the first is a constraint on the body rates and the second is a constraint on the total thrust. In this supplementary material, we focus on the second constraint on the maximum thrust [2] which is given as:

$$\ddot{x}^2 + \ddot{y}^2 + (\ddot{z} + g)^2 \leq f_{max}^2$$

where f_{max} is the maximum total thrust T that the quadrotor can produce. The discretized version of the constraint can be put in lifted form resulting in an inequality that is quadratic in $\hat{\mathbf{z}}$. We do this by first defining $\tilde{\mathbf{G}} \in \mathbb{R}^{3N}$ where $\tilde{\mathbf{G}}_{3k} = g \quad k = 1, \ldots, N$ and $\tilde{\mathbf{M}} \in \mathbb{R}^{3N \times 14N}$ where $\tilde{\mathbf{M}}_{3k+1,14k+3} = 1, \quad \tilde{\mathbf{M}}_{3k+2,14k+7} = 1, \quad \tilde{\mathbf{M}}_{3k+3,14k+11} = 1 \quad k = 0, \ldots, N-1$. Further after writing $\mathbf{F} \in \mathbb{R}^{3N}$ where $\mathbf{F}_k = f_{max}^2 \quad k = 1, \ldots, N$, we can rewrite our maximum thrust constraint as:

$$(\tilde{\mathbf{M}}\hat{\mathbf{z}} + \tilde{\mathbf{G}})^T(\tilde{\mathbf{M}}\hat{\mathbf{z}} + \tilde{\mathbf{G}}) < \mathbf{F}$$

.

Plugging in our expanded discretize model for $\hat{\mathbf{z}}$, we make the assumption that the quadratic coefficient $\hat{\mathbf{B}}^T\tilde{\mathbf{M}}^T\tilde{\mathbf{M}}\hat{\mathbf{B}}$ is relatively small. We justify this because this term contains squares of relatively small values in $\hat{\mathbf{B}}$, obtained through discretization of the linear flat model (14).

We, therefore, reduce the maximum thrust constraint to a linear constraint on the optimization variables $\hat{\mathbf{v}}$:

$$\mathbf{A_{con}}\hat{\mathbf{v}} \leq \mathbf{B_{con}}$$

where

$$\mathbf{A_{con}} = 2\mathbf{z_0}^T\hat{\mathbf{A}}^T\tilde{\mathbf{M}}^T\tilde{\mathbf{M}}\hat{\mathbf{B}} + 2\mathbf{G}^T\tilde{\mathbf{M}}\hat{\mathbf{B}}$$

and

$$\mathbf{B_{con}} = \mathbf{F} - \mathbf{G}^T\mathbf{G} - 2\mathbf{G}^T\tilde{\mathbf{M}}\hat{\mathbf{A}}\mathbf{z_0} - \mathbf{z_0}^T\hat{\mathbf{A}}^T\tilde{\mathbf{M}}^T\tilde{\mathbf{M}}\hat{\mathbf{A}}\mathbf{z_0}.$$

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