Recursive Estimation

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Problem Set:
Bayes Theorem,
recursive estimation using Bayes Theorem

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Notes:

- **Notation:** Unless otherwise noted, \( x, y, \) and \( z \) denote random variables, \( f_x(x) \) (or the shorthand \( f(x) \)) denotes the probability density function of \( x \), and \( f_{x|y}(x|y) \) (or \( f(x|y) \)) denotes the conditional probability density function of \( x \) conditioned on \( y \). The expected value is denoted by \( E[\cdot] \), the variance is denoted by \( \text{Var}[\cdot] \), and \( \text{Pr}(Z) \) denotes the probability that the event \( Z \) occurs. A normally distributed random variable \( x \) with mean \( \mu \) and variance \( \sigma^2 \) is denoted by \( x \sim \mathcal{N}(\mu, \sigma^2) \).

- Please report any errors found in this problem set to the teaching assistants (strimpe@ethz.ch or aschoellig@ethz.ch).
Problem Set

Problem 1
Mr. Jones has devised a gambling system for winning at roulette. When he bets, he bets on red, and places a bet only when the ten previous spins of the roulette have landed on a black number. He reasons that his chance of winning is quite large since the probability of eleven consecutive spins resulting in black is quite small. What do you think of this system?

Problem 2
Consider two boxes, one containing one black and one white marble, the other, two black and one white marble. A box is selected at random and a marble is drawn at random from the selected box. What is the probability that the marble is black?

Problem 3
In Problem 2, what is the probability that the first box was the one selected given that the marble is white?

Problem 4
Urn 1 contains two white balls and one black ball, while urn 2 contains one white ball and five black balls. One ball is drawn at random from urn 1 and placed in urn 2. A ball is then drawn from urn 2. It happens to be white. What is the probability that the transferred ball was white?

Problem 5
Stores A, B and C have 50, 75, 100 employees, and respectively 50, 60 and 70 percent of these are women. Resignations are equally likely among all employees, regardless of sex. One employee resigns and this is a woman. What is the probability that she works in store C?

Problem 6
a) A gambler has in his pocket a fair coin and a two-headed coin. He selects one of the coins at random, and when he flips it, it shows heads. What is the probability that it is the fair coin?

b) Suppose that he flips the same coin a second time and again it shows heads. What is now the probability that it is the fair coin?

c) Suppose that he flips the same coin a third time and it shows tails. What is now the probability that it is the fair coin?

Problem 7
Urn 1 has five white and seven black balls. Urn 2 has three white and twelve black balls. We flip a fair coin. If the outcome is heads, then a ball from urn 1 is selected, while if the outcome is tails, then a ball from urn 2 is selected. Suppose that a white ball is selected. What is the probability that the coin landed tails?
Problem 8
An urn contains \( b \) black balls and \( r \) red balls. One of the balls is drawn at random, but when it is put back in the urn \( c \) additional balls of the same color are put in with it. Now suppose that we draw another ball. Show that the probability that the first ball drawn was black given that the second ball drawn was red is \( \frac{b}{b + r + c} \).

Problem 9
Three prisoners are informed by their jailer that one of them has been chosen at random to be executed, and the other two are to be freed. Prisoner A asks the jailer to tell him privately which of his fellow prisoners will be set free, claiming that there would be no harm in divulging this information, since he already knows that at least one will go free. The jailer refuses to answer this question, pointing out that if A knew which of his fellows were to be set free, then his own probability of being executed would rise from \( \frac{1}{3} \) to \( \frac{1}{2} \), since he would then be one of two prisoners. What do you think of the jailer’s reasoning?

Problem 10
Let \( x \) and \( y \) be independent random variables. Let \( g(\cdot) \) and \( h(\cdot) \) be arbitrary functions of \( x \) and \( y \), respectively. Define the random variables \( v = g(x) \) and \( w = h(y) \). Prove that \( v \) and \( w \) are independent. That is, functions of independent random variables are independent.

Problem 11
Let \( x \) be a continuous, uniformly distributed random variable with \( x \in \mathcal{X} = [-5, 5] \). Let
\[
\begin{align*}
z_1 &= x + n_1 \\
z_2 &= x + n_2,
\end{align*}
\]
where \( n_1 \) and \( n_2 \) are continuous random variables with probability density functions
\[
\begin{align*}
f(n_1) &= \begin{cases} 
\alpha_1 (1 + n_1) & \text{for } -1 \leq n_1 \leq 0 \\
\alpha_1 (1 - n_1) & \text{for } 0 \leq n_1 \leq 1 \\
0 & \text{otherwise},
\end{cases} \\
f(n_2) &= \begin{cases} 
\alpha_2 \left(1 + \frac{1}{2}n_2\right) & \text{for } -2 \leq n_2 \leq 0 \\
\alpha_2 \left(1 - \frac{1}{2}n_2\right) & \text{for } 0 \leq n_2 \leq 2 \\
0 & \text{otherwise},
\end{cases}
\end{align*}
\]
where \( \alpha_1 \) and \( \alpha_2 \) are normalization constants. Assume that the random variables \( x, n_1, n_2 \) are independent, i.e. \( f(x, n_1, n_2) = f(x)f(n_1)f(n_2) \).

a) Calculate \( \alpha_1 \) and \( \alpha_2 \).

b) Calculate \( f(x|z_1 = 0, z_2 = 0) \).

c) Calculate \( f(x|z_1 = 0, z_2 = 1) \).

d) Calculate \( f(x|z_1 = 0, z_2 = 3) \).
Problem 12

Consider the following estimation problem: an object $B$ moves randomly on a circle with radius 1. The distance to the object can be measured from a given observation point $P$. The goal is to estimate the location of the object, see Figure 1.

The object $B$ can only move in discrete steps. The object’s location at time $k$ is given by $x(k) \in \{0, 1, \ldots, N - 1\}$, where

$$\theta(k) = 2\pi \frac{x(k)}{N}.$$

The dynamics are

$$x(k) = \text{mod} \left( x(k-1) + v(k), N \right), \quad k = 1, 2, \ldots,$$

where $v(k) = 1$ with probability $p$ and $v(k) = -1$ otherwise. Note that $\text{mod} \left( N, N \right) = 0$ and $\text{mod} \left( -1, N \right) = N - 1$. The distance sensor measures

$$z(k) = \left( (L - \cos \theta(k))^2 + (\sin \theta(k))^2 \right)^{\frac{1}{2}} + w(k),$$

where $w(k)$ represents the sensor error which is uniformly distributed on $[-e, e]$. We assume that $x(0)$ is uniformly distributed and $x(0)$, $v(k)$ and $w(k)$ are independent.

Simulate object movement and implement a Bayesian tracking algorithm that calculates for each time step $k$ the probability density function $f(x(k) | z(1 : k))$.

a) Test the following settings and discuss the results: $N = 100$, $x(0) = \frac{N}{4}$, $e = 0.5$,

$$L = 2, \quad p = 0.5,$$

$$L = 2, \quad p = 0.55,$$

$$L = 0.1, \quad p = 0.55,$$

$$L = 0, \quad p = 0.55.$$

b) How robust is the algorithm? Set $N = 100$, $x(0) = \frac{N}{4}$, $e = 0.5$, $L = 2$, $p = 0.55$ in the simulation, but use slightly different values for $p$ and $e$ in your estimation algorithm, $\hat{p}$ and $\hat{e}$, respectively. Test the algorithm and explain the result for:

$$\hat{p} = 0.45, \quad \hat{e} = e,$$

$$\hat{p} = 0.5, \quad \hat{e} = e,$$

$$\hat{p} = 0.9, \quad \hat{e} = e,$$

$$\hat{p} = p, \quad \hat{e} = 0.9,$$

$$\hat{p} = p, \quad \hat{e} = 0.45.$$
Sample solutions

Problem 1
We introduce a discrete random variable $x_i$, representing the outcome of the $i$th spin with $x_i \in \{\text{red, black}\}$. We assume that both are equally likely.

Assuming independence between spins, we have

$$\Pr(x_k, x_{k-1}, x_{k-2}, \ldots, x_1) = \Pr(x_k) \Pr(x_{k-1}) \ldots \Pr(x_1).$$

The probability of eleven consecutive spins resulting in black is

$$\Pr(x_{11} = \text{black}, x_{10} = \text{black}, x_9 = \text{black}, \ldots, x_1 = \text{black}) = \left(\frac{1}{2}\right)^{11}.$$

This value is actually quite small. However, given that the previous ten were black, we calculate

$$\Pr(x_{11} = \text{black}|x_{10} = \text{black}, x_9 = \text{black}, \ldots, x_1 = \text{black}) = \frac{1}{2},$$

i.e. it is equally likely that ten black spins in a row are followed by a red spin, as that they are followed by another black spin (by the independence assumption).

Example: Consider two spins. All combinations (red, red), (red, black), (black, red), (black, black) are equally likely. Consequently, $\Pr(x_2 = \text{black}|x_1 = \text{red}) = \frac{1}{2}$, $\Pr(x_2 = \text{red}|x_1 = \text{red}) = \frac{1}{2}$, $\Pr(x_2 = \text{red}|x_1 = \text{black}) = \frac{1}{3}$.

Therefore, this strategy is bad :)

Problem 2
We introduce two random variables. Let $x \in \{1, 2\}$ represent which box is chosen (box 1 or 2) with probability $f_x(1) = f_x(2) = \frac{1}{2}$. Furthermore, let $y \in \{b, w\}$ represent the color of the drawn marble, where $b$ is a black and $w$ a white marble with probabilities

$$f_{y|x}(b|1) = f_{y|x}(w|1) = \frac{1}{2},$$

$$f_{y|x}(b|2) = \frac{2}{3}, \quad f_{y|x}(w|2) = \frac{1}{3}.$$

Then, by the total probability theorem, we have

$$f_y(b) = f_{y|x}(b|1) f_x(1) + f_{y|x}(b|2) f_x(2) = \frac{1}{2} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} = \frac{7}{12}.$$

Problem 3

$$f_{x|y}(1|w) = \frac{f_{y|x}(w|1) f_x(1)}{f_y(w)} = \frac{\frac{1}{2} \cdot \frac{1}{2}}{1 - f_y(b)} = \frac{\frac{1}{4}}{\frac{5}{12}} = \frac{3}{5}.$$
Problem 4

Let \( x \in \{b, w\} \) represent the color of the ball drawn from urn 1, where \( f_x(b) = \frac{1}{3}, f_x(w) = \frac{2}{3} \), and \( y \in \{b, w\} \) be the color of the ball subsequently drawn from urn 2. Considering the different possibilities, we have

\[
\begin{align*}
    f_{y|x}(b|b) &= \frac{6}{7}, & f_{y|x}(b|w) &= \frac{5}{7}, \\
    f_{y|x}(w|b) &= \frac{1}{7}, & f_{y|x}(w|w) &= \frac{2}{7}.
\end{align*}
\]

We now need to calculate \( f_{x|y}(w|w) \),

\[
\begin{align*}
    f_{x|y}(w|w) &= \frac{f_{y|x}(w|w) \cdot f_x(w)}{f_x(w)} = \frac{\frac{2}{7} \cdot \frac{2}{3}}{\frac{2}{7} \cdot \frac{1}{3} + \frac{2}{7} \cdot \frac{2}{3}} = \frac{4}{5}.
\end{align*}
\]

Problem 5

We introduce two discrete random variables, \( x \in \{A, B, C\} \) and \( y \in \{M, F\} \), where \( x \) represents which store an employee works in, and \( y \) the sex of the employee. From the problem description, we have

\[
\begin{align*}
    f_x(A) &= \frac{50}{225} = \frac{2}{9}, \\
    f_x(B) &= \frac{75}{225} = \frac{1}{3}, \\
    f_x(C) &= \frac{100}{225} = \frac{4}{9},
\end{align*}
\]

and the probability that an employee is a woman is

\[
\begin{align*}
    f_{y|x}(F|A) &= \frac{1}{2}, \\
    f_{y|x}(F|B) &= \frac{3}{5}, \\
    f_{y|x}(F|C) &= \frac{7}{10}.
\end{align*}
\]

To answer the question, we need to calculate \( f_{x|y}(C|F) \),

\[
\begin{align*}
    f_{x|y}(C|F) &= \frac{f_{y|x}(F|C) \cdot f_x(C)}{f_y(F)} = \frac{\frac{7}{10} \cdot \frac{4}{9}}{\sum_{i \in \{A,B,C\}} f_{y|x}(F|i) \cdot f_x(i)} = \frac{\frac{7}{10} \cdot \frac{4}{9}}{\frac{1}{2} \cdot \frac{2}{9} + \frac{3}{5} \cdot \frac{1}{3} + \frac{7}{10} \cdot \frac{4}{9}} = \frac{1}{2}.
\end{align*}
\]

Problem 6

a) Let \( x \in \{F, U\} \) represent whether it is a fair (\( F \)) or an unfair (\( U \)) coin with \( f_x(F) = f_x(U) = \frac{1}{2} \). We introduce \( y \in \{h, t\} \) to represent how the toss comes up (heads or tails) with

\[
\begin{align*}
    f_{y|x}(h|F) &= \frac{1}{2}, & f_{y|x}(t|F) &= \frac{1}{2}, \\
    f_{y|x}(h|U) &= 1, & f_{y|x}(t|U) &= 0.
\end{align*}
\]
To answer the question, we have to calculate $f_{x|y}(F|h)$,

$$
f_{x|y}(F|h) = \frac{f_{y|x}(h|F) \cdot f_x(F)}{f_y(h)} = \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} + 1} = \frac{1}{3}.
$$

b) Let $y_1$ represent the result of the first flip and $y_2$ that of the second flip. We assume independence between flips (if conditioned on $x$), yielding $f_{y_1,y_2|x}(y_1,y_2|x) = f_{y_1|x}(y_1|x) \cdot f_{y_2|x}(y_2|x)$.

The question asks us to calculate $f_{x|y_1,y_2}(F|h,h)$,

$$
f_{x|y_1,y_2}(F|h,h) = \frac{f_{y_1,y_2|x}(h,h|F) \cdot f_x(F)}{f_{y_1,y_2|(h,h)}} = \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} + 1} = \frac{5}{15}.
$$

c) Obviously, the probability then is 1.

**Problem 7**

We introduce the following two random variables:

- $x \in \{1, 2\}$ represents which box is chosen (box 1 or 2) with probability $f_x(1) = f_x(2) = \frac{1}{2}$,

- $y \in \{b, w\}$ represents the color of the drawn ball, where $b$ is a black and $w$ a white ball with probabilities

$$
\begin{align*}
    f_{y|x}(b|1) &= \frac{7}{12}, & f_{y|x}(w|1) &= \frac{5}{12}, \\
    f_{y|x}(b|2) &= \frac{12}{15}, & f_{y|x}(w|2) &= \frac{3}{15}.
\end{align*}
$$

To answer the question, we need to calculate $f_{x|y}(2|w)$,

$$
f_{x|y}(2|w) = \frac{f_{y|x}(w|2) \cdot f_x(2)}{f_y(w)} = \frac{\frac{3}{15} \cdot \frac{1}{2} \cdot \frac{1}{2}}{\frac{3}{15} \cdot \frac{1}{2} + \frac{1}{2}} = \frac{12}{37}.
$$

**Problem 8**

Let $x_1 \in \{B, R\}$ be the color (black or red) of the first ball drawn with $f_{x_1}(B) = \frac{b}{b+r+c}$ and $f_{x_1}(R) = \frac{b}{b+r+c}$; and let $x_2 \in \{B, R\}$ be the color of the second ball with

$$
\begin{align*}
    f_{x_2|x_1}(B|R) &= \frac{b}{b+r+c}, & f_{x_2|x_1}(R|R) &= \frac{c+r}{b+r+c} \\
    f_{x_2|x_1}(B|B) &= \frac{b+c}{b+r+c}, & f_{x_2|x_1}(R|B) &= \frac{r}{b+r+c}.
\end{align*}
$$
The answer to the question is now,

\[
f_{x_1|x_2}(B|R) = \frac{f_{x_2|x_1}(R|B) f_{x_1}(B)}{f_{x_2}(R)} = \frac{\frac{r}{b+r+c} \cdot \frac{b}{b+r}}{\frac{b}{b+r} + \frac{r}{b+r} \cdot \frac{b}{b+r}} = \frac{b}{b+r+c}.
\]

**Problem 9**

We can approach the solution in two ways:

**Popular, descriptive solution.** The probability that A is to be executed is \(\frac{1}{3}\), and there is a chance of \(\frac{2}{3}\) that one of the others was chosen. If the jailer gives away the name of one of the fellow prisoners who will be set free, prisoner A does not get new information about his own fate, but the probability of the remaining prisoner (B or C) to be executed is now \(\frac{2}{3}\). The probability of A being executed is still \(\frac{1}{3}\).

**Bayesian analysis.** Let \(x\) represent which prisoner is to be executed, where \(x \in \{A, B, C\}\). We assume that it is a random choice, i.e. \(f_x(A) = f_x(B) = f_x(C) = \frac{1}{3}\).

Now let \(y \in \{B, C\}\) be the prisoner name given away by the jailer. We can now write the conditional probabilities:

\[
f_{y|x}(y|x) = \begin{cases} 
0 & \text{if } x = y \text{ (the jailer does not lie)} \\
\frac{1}{2} & \text{if } x = A \text{ (A is to be executed, jailer mentions B and C with equal probability)} \\
1 & \text{if } x \neq A \text{ (jailer is forced to give the name of the other prisoner to be set free)}
\end{cases}
\]

You could also do this with a table of the form:

| \(y\) | \(x\) | \(f_{y|x}(y|x)\) |
|-------|-------|--------------------|

To answer the question, we have to compare \(f_{x|y}(A|\bar{y})\), \(\bar{y} \in \{B, C\}\), with \(f_x(A)\):

\[
f_{x|y}(A|\bar{y}) = \frac{f_{y|x}(\bar{y}|A) f_x(A)}{f_{y}(\bar{y})} = \sum_{k \in \{A,B,C\}} \frac{\frac{1}{2} \cdot \frac{1}{3} \cdot f_{y|x}(\bar{y}|k) \cdot f_x(k)}{f_{y}(\bar{y})} = \frac{1}{3},
\]

where (not \(\bar{y}\)) = C if \(\bar{y} = B\) and (not \(\bar{y}\)) = B if \(\bar{y} = C\). The value of the posterior probability is the same as the prior one \(f_x(A)\). The jailer is wrong: prisoner A gets no additional information from the jailer about his own fate!

See also Wikipedia: Three prisoners problem, Monty Hall problem.
Problem 10

Consider the joint cumulative distribution

\[ F_{v,w}(\bar{v}, \bar{w}) = \Pr ((v \leq \bar{v}) \text{ and } (w \leq \bar{w})) \]

\[ = \Pr ((g(x) \leq \bar{v}) \text{ and } (h(y) \leq \bar{w})) . \]

We define the sets \( A_{\bar{v}} \) and \( A_{\bar{w}} \) as below:

\[ A_{\bar{v}} = \{ x \in \bar{X} : g(x) \leq \bar{v} \} \]

\[ A_{\bar{w}} = \{ y \in \bar{Y} : h(y) \leq \bar{w} \} . \]

Now we can deduce

\[ F_{v,w}(\bar{v}, \bar{w}) = \Pr ((x \in A_{\bar{v}}) \text{ and } (y \in A_{\bar{w}})) \]

\[ = \Pr (x \in A_{\bar{v}}) \Pr (y \in A_{\bar{w}}) \text{ (by independence assumption)} \]

\[ = \Pr (v \leq \bar{v}) \Pr (w \leq \bar{w}) \]

\[ = F_v(\bar{v}) F_w(\bar{w}) . \]

Therefore \( v \) and \( w \) are independent.

Problem 11

a) We do this by integrating the probability density functions (refer to Figure 2).

For \( n_1 \), we obtain

\[ \int_{-\infty}^{\infty} f_{n_1}(\bar{n}_1) d\bar{n}_1 = \alpha_1 \left( \int_{-1}^{0} (1 + \bar{n}_1) d\bar{n}_1 + \int_{0}^{1} (1 - \bar{n}_1) d\bar{n}_1 \right) \]

\[ = \alpha_1 \left( 1 - \frac{1}{2} + 1 - \frac{1}{2} \right) = 1. \]

Therefore \( \alpha_1 = 1 \).

For \( n_2 \), we get

\[ \int_{-\infty}^{\infty} f_{n_2}(\bar{n}_2) d\bar{n}_2 = \alpha_2 \left( \int_{-2}^{0} \left( 1 + \frac{1}{2} \bar{n}_2 \right) d\bar{n}_2 + \int_{0}^{2} \left( 1 - \frac{1}{2} \bar{n}_2 \right) d\bar{n}_2 \right) \]

\[ = \alpha_2 (2 - 1 + 2 - 1) = 2\alpha_2 = 1. \]

Therefore \( \alpha_2 = \frac{1}{2} \).

Figure 2: Determining \( \alpha_1 \) and \( \alpha_2 \).
b) All values $x$ are equally likely (state of maximum entropy),

$$f_{x}(x) = \begin{cases} 
\frac{1}{10} & \text{for } -5 \leq x \leq 5 \\
0 & \text{otherwise.} 
\end{cases}$$

As a sanity check, we can confirm that $\int_{-\infty}^{\infty} f_{x}(\bar{x}) \, d\bar{x} = 1$.

By the independence assumption, we have

$$f_{z_1,z_2|x}(z_1, z_2|x) = f_{z_1|x}(z_1|x) f_{z_2|x}(z_2|x).$$

Now we calculate the observation likelihoods $f_{z_1|x}(z_1|x)$ and $f_{z_2|x}(z_2|x)$, noting $z_1 - x = n_1$, $z_2 - x = n_2$ and that $x$ is given/fixed. First, we compute

$$f_{z_1|x}(z_1|x) = \begin{cases} 
\alpha_1 (1 - z_1 + x) & \text{for } 0 \leq z_1 - x \leq 1 \\
\alpha_1 (1 + z_1 - x) & \text{for } -1 \leq z_1 - x \leq 0 \\
0 & \text{otherwise.} 
\end{cases}$$

The PDF is illustrated in Figure 3.

![Figure 3: Illustration of the observation likelihood $f_{z_1|x}(z_1|x)$.](image)

Similarly, for $f_{z_2|x}(z_2|x)$,

$$f_{z_2|x}(z_2|x) = \begin{cases} 
\alpha_2 (1 - \frac{1}{2}z_2 + \frac{1}{2}x) & \text{for } 0 \leq z_2 - x \leq 2 \\
\alpha_2 (1 + \frac{1}{2}z_2 - \frac{1}{2}x) & \text{for } -2 \leq z_2 - x \leq 0 \\
0 & \text{otherwise.} 
\end{cases}$$

We now invoke Bayes’ theorem:

$$f_{x|z_1,z_2}(x|z_1, z_2) = \frac{\int_{-5}^{5} f_{z_1|x}(z_1|x) f_{z_2|x}(z_2|x) f_{x}(x) \, d\bar{x}}{\int_{-5}^{5} f_{z_1|x}(z_1|\bar{x}) f_{z_2|x}(z_2|\bar{x}) f_{x}(\bar{x}) \, d\bar{x}},$$

where $n(z_1, z_2)$ is the normalization.

Given $z_1 = 0$, $z_2 = 0$, we can write the numerator as $\text{num} \,(x)$,

$$\text{num} \,(x) = \frac{1}{10} f_{z_1|x}(0|x) f_{z_2|x}(0|x).$$

We consider four different intervals of $x$: $[-5, -1]$, $[-1, 0]$, $[0, 1]$ and $[1, 5]$. Evaluating $\text{num} \,(x)$ for these intervals results in:
• for $x \in [-5, -1]$ or $x \in [1, 5],
\quad \text{num}(x) = 0$
• for $x \in [-1, 0],
\quad \text{num}(x) = \frac{1}{10} \alpha_1 (1 + x) \alpha_2 \left(1 + \frac{x}{2}\right) = \frac{1}{20} (1 + x) \left(1 + \frac{x}{2}\right)$
• for $x \in [0, 1],
\quad \text{num}(x) = \frac{1}{10} \alpha_1 (1 - x) \alpha_2 \left(1 - \frac{x}{2}\right) = \frac{1}{20} (1 - x) \left(1 - \frac{x}{2}\right)$.

Therefore,
\[ f_{x|z_1, z_2}(x|0, 0) = \frac{1}{n(0, 0)} \text{num}(x) \]
as illustrated in Figure 4.

\[ f_{x|z_1, z_2}(x|0, 0) = \begin{cases} 
0 & \text{for } -5 \leq x \leq -1, \ 1 \leq x \leq 5 \\
\frac{12}{10} (1 + x) \left(1 + \frac{x}{2}\right) & \text{for } -1 \leq x \leq 0 \\
\frac{12}{10} (1 - x) \left(1 - \frac{x}{2}\right) & \text{for } 0 \leq x \leq 1 
\end{cases} \]

We see that this is symmetric about $x = 0$ with a maximum at $x = 0$ – both sensors “agree”.

c) Similar to the above. Given $z_1 = 0, z_2 = 1$, we write the numerator as
\[ \text{num}(x) = \frac{1}{10} f_{z_1|x}(0|x) f_{z_2|x}(1|x). \] (2)
for \( x \in [-5, -1] \) or \( x \in [1, 5] \),
\[
\text{num}(x) = 0
\]
for \( x \in [-1, 0] \),
\[
\text{num}(x) = \frac{1}{10} \alpha_1 (1 + x) \alpha_2 \left( \frac{1}{2} + \frac{1}{2} x \right) = \frac{1}{40} (1 + x)^2
\]
for \( x \in [0, 1] \),
\[
\text{num}(x) = \frac{1}{10} \alpha_1 (1 - x) \alpha_2 \left( \frac{1}{2} + \frac{1}{2} x \right) = \frac{1}{40} (1 - x^2).
\]
Normalizing yields
\[
n(0, 1) = \int_{-1}^{1} \text{num}(x) \, dx = \frac{1}{120} + \frac{2}{120} = \frac{1}{40}.
\]
We now have the solution as
\[
f_{x|z_1, z_2}(x|0, 1) = \begin{cases} 
0 & \text{for } -5 \leq x \leq -1, \ 1 \leq x \leq 5 \\
(1 + x)^2 & \text{for } -1 \leq x \leq 0 \\
1 - x^2 & \text{for } 0 \leq x \leq 1.
\end{cases}
\]
The solution is depicted in Figure 5.

The probability values are higher for positive \( x \) values because of the measurement \( z_2 = 1 \).

\( \text{d) } \) We start in the same fashion: given \( z_1 = 0 \) and \( z_2 = 3 \),
\[
\text{num}(x) = \frac{1}{10} f_{z_1|x}(0|x) f_{z_2|x}(3|x).
\]
However, the intervals of positive probability of \( f_{z_1|x}(0|x) \) and \( f_{z_2|x}(3|x) \) do not overlap, i.e.
\[
\text{num}(x) = 0 \quad \forall \ x \in [-5, 5].
\]
In other words, given our noise model for \( n_1 \) and \( n_2 \), there is no chance to measure \( z_1 = 0 \) and \( z_2 = 3 \). Therefore, \( f_{x|z_1, z_2}(x|0, 3) \) is not defined.
Problem 12

The Matlab code is available on the class webpage. We notice the following:

a) • bimodal distribution
   • decay
   • works
   • uniform distribution \( \forall k = 1, 2, \ldots \)

b) • wrong result: estimation vs. real position
   • bimodal – we cannot differentiate
   • incorrect assumption
   • washes out
   • crashes! Think about why . . .