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**Quiz** **April 14th, 2010**

# Introduction to Recursive Filtering and Estimation

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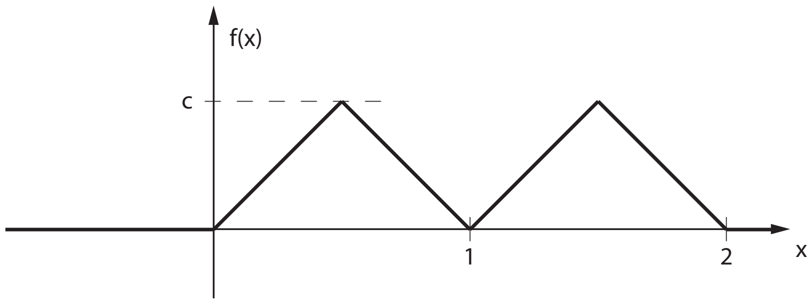
# Solutions

| | |
|---|---|
| **Duration:** | **45 minutes** |
| **Number of Problems:** | **4** |
| **Permitted Aids:** | None.<br>Use only the prepared sheets for your solutions. |

## Problem 1 25%

Consider the following function $f(x)$:




*Please write your answers in the provided boxes.*

**a)** Choose the constant $c$ such that $f(x)$ is a valid probability density function for the continuous random variable $x$.

$c =$

**b)** Determine the expected value $\mathrm{E}[x]$ for the given probability density function $f(x)$.

$\mathrm{E}[x] =$

*Please circle either 'Yes' or 'No' in the questions below.*

The following statements refer to the given probability density function $f(x)$.

**c)** The expected value is the maximum likelihood estimate.

Yes No

**d)** The variance $\mathrm{Var}(x)$ is less than 1.

Yes No

**e)** The variance $\mathrm{Var}(x - 0.5) = \mathrm{Var}(x)$.

Yes No

**Note:** *For questions c)-e) you get: +5 % for a correct answer, -5 % for an incorrect answer and 0 % for no answer. If you change your mind, please cross out all options ('Yes' and 'No') and write either 'Yes' or 'No' alongside, or leave it blank.*

## Solution 1

**a)** $\mathbf{c = 1}$.

In order for $f(x)$ to be a valid probability density function, the following property has to be satisfied:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1\,.$$

That is, the constant $c$ has to be chosen such that the area under the given function $f(x)$ is unity. By symmetry,

$$\int_{-\infty}^{\infty} f(x)\,dx = 4\int_{0}^{0.5} f(x)\,dx = 4\,A_{\text{triangle}} = c\cdot 1 = 1\,,$$

where the area of the four triangles $4\,A_{\text{triangle}}$ result in a rectangle with side lengths of $c$ and 1. Thus, the constant $c$ is chosen to be $c = 1$.

**b)** $\mathrm{E}[\mathbf{x}] = \mathbf{1}$.

Intuitively, the expected value can be obtained by viewing the probability density function $f(x)$ as a continuous mass distribution along a line and determining the corresponding center of mass.

When applying the definition of the expected value,

$$\mathrm{E}[x] = \int_{-\infty}^{\infty} x f(x)\,dx\,,$$

the symmetry of the given function $f(x)$ can be exploited by shifting the function to the left and introducing $\xi = x - 1$,

$$\begin{aligned}\mathrm{E}[x] &= \int_{0}^{2} x f(x)\,dx \\ &= \int_{-1}^{1} (\xi+1)\,f(\xi+1)\,d\xi = \int_{-1}^{1} \xi\,g(\xi)\,d\xi + \int_{-1}^{1} g(\xi)\,d\xi = 0 + 1 = 1\,,\end{aligned}$$

where $g(\xi) = f(\xi+1)$ represents the left-shifted function. We use the fact that $\xi\,g(\xi)$ is an odd function and

$$\int_{-1}^{1} g(\xi)\,d\xi = 1$$

was calculated in part a).

**c)** **No.**

The maximum likelihood estimate is given by

$$\hat{x}_{ML} = \arg\max_{x} f(x) = \{0.5,\, 1.5\}\,,$$

whereas the expected value $\mathrm{E}[x] = 1$.

**d)** **Yes.**

In the scalar case, the variance is defined as

$$\mathrm{Var}(x) = \mathrm{E}[(x-\bar{x})^2]\,,$$

where $\bar{x} = \mathrm{E}[x]$. For the given probability density function, that is

$$\mathrm{Var}(x) = \int_0^2 \underbrace{(x-1)^2}_{\leq 1} f(x)\,dx \leq \int_0^2 f(x)\,dx = \mathrm{E}[x] = 1\,.$$

Note that the two cases, $x = 0$ and $x = 2$, where $(x-1)^2 = 1$, have no contribution since $f(0) = f(2) = 0$. Thus, the variance $\mathrm{Var}(x)$ is strictly smaller than 1.

**e)** **Yes.**

For any constant value $b$, the following relation holds:

$$\mathrm{E}[x-b] = \mathrm{E}[x] - b \qquad \text{(law of the unconscious statistician)}\,.$$

Thus,

$$\mathrm{Var}(x-b) = \mathrm{E}\left[\left(x - b - \left(\mathrm{E}\left[x\right] - b\right)\right)^2\right] = \mathrm{E}\left[\left(x - \mathrm{E}\left[x\right]\right)^2\right] = \mathrm{Var}(x)\,.$$

## Problem 2 25%

*Please circle either 'Yes' or 'No' in the following questions.*

Consider the joint probability density function $f(x, y)$ for the continuous random variables $x$ and $y$:

$$f(x, y) = 6x^2y, \qquad x, y \in [0, 1] \ .$$

**a)** The probability density function of $x$ is $f(x) = x^2$.

Yes No

**b)** The cumulative distribution function of $y$ is $F(y) = y^2$.

Yes No

**c)** The random variables $x$ and $y$ are independent.

Yes No

**d)** The conditional probability density function of $x$ conditioned on $y$ is $f(x|y) = x^2$.

Yes No

**e)** The probability of $x \leq 1$ and $y \leq 0.5$ is $\Pr(x \leq 1, y \leq 0.5) = 0.25$.

Yes No

**Note:** *For each question you get: +5 % for a correct answer, -5 % for an incorrect answer and 0 % for no answer. If you change your mind, please cross out all options ('Yes' and 'No') and write either 'Yes' or 'No' alongside, or leave it blank.*

## Solution 2

**a) No.**

The desired probability density function $f(x)$ is obtained by marginalization of the joint probability density function $f(x,y)$,

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 f(x,y)\,dy = 3x^2\,. \tag{1}$$

**b) Yes.**

Similar to a), we calculate

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 f(x,y)\,dx = 2y\,. \tag{2}$$

The cumulative distribution function $F(y)$ is given by

$$F(y) = \int_{-\infty}^{y} f(\tilde{y})\,d\tilde{y} = y^2, \qquad y \in [0,1]\,. \tag{3}$$

**c) Yes.**

The random variables $x$ and $y$ are said to be independent if

$$f(x,y) = f(x)f(y)\,.$$

With (1) and (2), the independence for the given function $f(x,y)$ is shown,

$$f(x)f(y) = 3x^2 \cdot 2y = 6x^2y = f(x,y)\,.$$

**d) No.**

Because of the independence of $x$ and $y$ shown above, we obtain

$$f(x|y) = f(x) = 3x^2\,.$$

**e) Yes.**

Recalling the independence of $x$ and $y$, Equation (3) and $x, y \in [0,1]$, we get

$$Pr(x \leq 1, y \leq 0.5) = F_{x,y}(1, 0.5) = F_x(1)F_y(0.5) = 1 \cdot (0.5)^2 = 0.25\,.$$

## Problem 3 25%

A bag contains twelve coins: four coins are fair, six coins are biased and show tails with a probability 2/3, and two coins are two-headed. A coin is chosen at random from the bag. The coin is tossed. It happens to be heads.

What is the probability that the chosen coin was a two-headed one?

## Solution 3

We introduce the random variable $x \in \{r, b, w\}$ representing the outcome of choosing a coin at random from the bag. A fair coin is denoted by $r$, a biased coin by $b$, and $w$ denotes a two-headed coin. The probabilities are as follows:

$$f_x(r) = \frac{1}{3}, \qquad f_x(b) = \frac{1}{2}, \qquad f_x(w) = \frac{1}{6}.$$

A second random variable $y \in \{h, t\}$ is introduced representing the outcome of flipping a coin, which can be either heads, $y = h$, or tails, $y = t$. When tossing the previously chosen coin, the conditional probabilities of obtaining heads are

$$f_{y|x}(h|r) = \frac{1}{2}, \qquad f_{y|x}(h|b) = \frac{1}{3}, \qquad f_{y|x}(h|w) = 1.$$

Bayes Theorem yields the desired value,

$$f_{x|y}(w|h) = \frac{f_{y|x}(h|w)\, f_x(w)}{f_y(h)},$$

where

$$f_y(h) = f_{y|x}(h|w)\, f_x(w) + f_{y|x}(h|b)\, f_x(b) + f_{y|x}(h|r)\, f_x(r)$$

with the total probability theorem. Given the numerical values above, we get

$$f_{x|y}(w|h) = \frac{1}{3}.$$

Note that $f_{x|y}(w|h) = f_{x|y}(b|h) = f_{x|y}(r|h)$.

## Problem 4 25%

Consider the following estimation problem: an object $B$ moves randomly on a circle with radius 1. From a given observation point $P$ at location $x = L$, the angle $\phi$ can be measured, which describes the angle from the horizontal line to the line segment connecting $P$ and $B$, see Figure 1. The goal is to estimate the location of the object $B$.

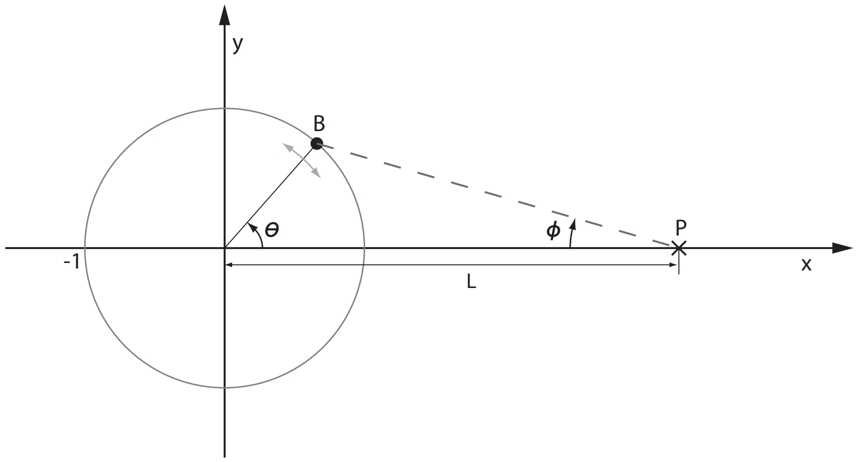



*Figure 1*

**a)** The object $B$ can only move in discrete steps. The object's location at time $k$ is given by $x(k) \in \{0, 1, \ldots, N-1\}$, where

$$\theta(k) = 360\,\frac{x(k)}{N}.$$

The angle $\theta(k)$ is in degrees. The dynamics are

$$x(k) = \mod(x(k-1) + v(k)\,,\, N), \qquad k = 1, 2, \ldots.$$

with $v(k) \in \{-1, 0, 1\}$. The object $B$ moves clockwise, $v(k) = -1$, with probability $p$, it moves counter-clockwise, $v(k) = 1$, with probability $p$, and stays at its location, $v(k) = 0$, with probability $(1 - 2p)$, $p < 0.5$.
Note that $\mod(N, N) = 0$ and $\mod(-1, N) = N - 1$.

Determine the conditional probability density function $f(x(k)|x(k-1))$.

**b)** The angle sensor sitting at point $P$ returns a measurement $z$ of the angle $\phi$ corrupted by an additive noise $w$, that is

$$z = \phi + w\,.$$

The sensor noise $w$ has the following probability density function:

$$f(w) = \begin{cases} \frac{1}{2}\left(1 + \frac{1}{2}w\right) & \text{for } -2 \le w \le 0 \\ \frac{1}{2}\left(1 - \frac{1}{2}w\right) & \text{for } \quad 0 \le w \le 2 \\ 0 & \text{otherwise}\,. \end{cases}$$

The angle $\phi$ and the noise $w$ are in degrees.

- Write $\phi$ as a function of $\theta$.
- Determine the conditional probability density function $f(z|\theta)$.

## Solution 4

**a)** The conditional probability $f(x(k)|x(k-1))$ is directly derived from the object's dynamics,

$$f(x(k)|x(k-1)) = \begin{cases} p & \text{if} \quad x(k) = \mod(x(k-1)+1\,,\, N) \\ p & \text{if} \quad x(k) = \mod(x(k-1)-1\,,\, N) \\ 1-2p & \text{if} \quad x(k) = x(k-1) \\ 0 & \text{otherwise}\,. \end{cases}$$

**b)** The position of object $B$ is given by its coordinates $(x_B, y_B)$, which can be expressed in terms of $\theta$,

$$x_B = \cos\theta, \qquad y_B = \sin\theta\,.$$

For the angle $\phi$, the following relation is derived

$$\tan\phi = \frac{y_B}{L - x_B} = \frac{\sin\theta}{L - \cos\theta}\,.$$

Thus,

$$\begin{aligned} \phi &= \arctan\left(\frac{\sin\theta}{L-\cos\theta}\right) \\ &= h(\theta)\,, \end{aligned}$$

where the abbreviation $h(\cdot)$ is used in the following derivations.

Given the angle $\theta$, the conditional probability density function of $z$ can be determined using the relationship

$$w = z - \phi = z - h(\theta)$$

and the given probability distribution of $w$, resulting in

$$f(z|\theta) = \begin{cases} \frac{1}{2}\Big(1 + \frac{1}{2}\big(z - h(\theta)\big)\Big) & \text{for} \quad -2 \leq z - h(\theta) \leq 0 \\ \frac{1}{2}\Big(1 - \frac{1}{2}\big(z - h(\theta)\big)\Big) & \text{for} \quad 0 \leq z - h(\theta) \leq 2 \\ 0 & \text{otherwise}\,. \end{cases}$$