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Introduction to Recursive Filtering and Estimation

Spring 2010

Problem Set: Introduction to estimation

Notes:

- **Notation:** Unless otherwise noted, x, y, and z denote random variables, $f_x(x)$ (or the short hand $f(x)$) denotes the probability density function of x, and $f_{x|y}(x|y)$ (or $f(x|y)$) denotes the conditional probability density function of x conditioned on y. The expected value is denoted by $\mathrm{E}[\cdot]$, the variance is denoted by $\mathrm{Var}(\cdot)$ and $\Pr(Z)$ denotes the probability that the event Z occurs. A normally distributed random variable x with mean μ and variance σ^2 is denoted by $x \sim \mathcal{N}(\mu, \sigma^2)$.
- Please report any error that you may find to the teaching assistants (`strimpe@ethz.ch` or `aschoellig@ethz.ch`).

Problem Set

Problem 1

Find the maximum likelihood (ML) estimate $\hat{x}_{ML} := \arg\max_x f(z|x)$ when

$$z_i = x + w_i\,, \qquad i = 1, 2\,,$$

where w_i are assumed to be independent with $w_i \sim \mathcal{N}(0, \sigma_i^2)$ and $z_i, w_i, x \in \mathbb{R}$.

Problem 2

Solve Problem 1 for independent w_i uniformly distributed on $[-1, 1]$.

Problem 3

Find the ML estimate $\hat{x}_{ML}$ when

$$z = Hx + w\,,$$

where $x \in \mathbb{R}^n$ and $z, w \in \mathbb{R}^m$, $n \leq m$. The matrix H has full rank.
The components of the vector w are assumed to be independent with $w_i \sim \mathcal{N}(0, \sigma_i^2)$, $i = 1, \ldots, m$.

How does this relate to weighted least squares?

Problem 4

Find the maximum a posteriori (MAP) estimate $\hat{x}_{MAP} := \arg\max_x f(z|x)f(x)$ when

$$z = x + w\,, \qquad x, z, w \in \mathbb{R}\,,$$

where x is exponentially distributed,

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

with $\lambda = 1$ and $w \sim \mathcal{N}(0,\, 1)$.

Problem 5

Prove that

$$\frac{\partial\ \mathrm{Tr}(ABA^T)}{\partial A} = 2AB \qquad \text{if } \ B = B^T,$$

for $A, B \in \mathbb{R}^{2\times 2}$, where $\mathrm{Tr}(\cdot)$ denotes the trace of a matrix.

Problem 6

Prove that

$$\frac{\partial\ \mathrm{Tr}(AB)}{\partial A} = B^T,$$

for $A, B \in \mathbb{R}^{2\times 2}$.

Problem 7

Apply the recursive least squares algorithm when

$$z(k) = x + w(k)\,, \qquad k = 1, 2, \dots\,,$$

where

$$\mathrm{E}[x] = 0,\;\; \mathrm{Var}(x) = 1, \quad \text{and} \qquad \mathrm{E}[w(k)] = 0,\;\; \mathrm{Var}\,(w(k)) = 1. \tag{1}$$

The random variables $\{x, w(1), w(2), \dots\}$ are assumed to be independent and $z(k), w(k), x \in \mathbb{R}$.

a) Solve for $K(k)$ and $P(k)$.

b) Simulate the algorithm for normally distributed continuous random variables $w(k)$ and x with the above properties (1). Choose k up to 10000.

c) Simulate the algorithm for x and $w(k)$ being discrete random variables that take the value 1 and -1 with equal probability. Note that x and $w(k)$ satisfy (1). Choose k up to 10000.

d) For the distribution in part c), what is the optimal minimum mean-squared error (MMSE) estimate?

Problem Set 3

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Problem 1

$f(w_i) \propto \exp\left(-\frac{1}{2}\frac{w_i^2}{\sigma_i^2}\right)$

$\uparrow$ proportional

$f(z_1, z_2 \mid x) = f(z_1|x)\, f(z_2|x)$ (conditional independent)

$\propto \exp\left(-\frac{1}{2}\left(\frac{(z_1-x)^2}{\sigma_1^2} + \frac{(z_2-x)^2}{\sigma_2^2}\right)\right)$

$\Rightarrow$ Differentiate with respect to x and set to 0:

$\frac{(z_1-\hat{x})}{\sigma_1^2} + \frac{(z_2-\hat{x})}{\sigma_2^2} \overset{!}{=} 0 \iff \hat{x} = \frac{z_1\sigma_2^2 + z_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}$

weighted sum!

Interpretation:

$\sigma_1^2 = 0 \;:\; \hat{x} = z_1$

$\sigma_2^2 = 0 \;:\; \hat{x} = z_2$

Problem 2

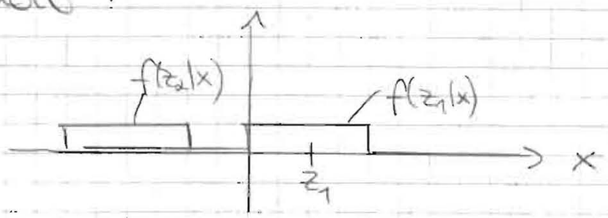
$$w_i = \begin{cases} 1/2 & w_i \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$

$$f(z_i|x) = \begin{cases} 1/2 & \text{for } -1 \le z_i - x \le 1 \text{ or } z_i - 1 \le x \le z_i + 1 \\ 0 & \text{otherwise} \end{cases}$$

$f(z_1, z_2|x) = f(z_1|x)\, f(z_2|x)$

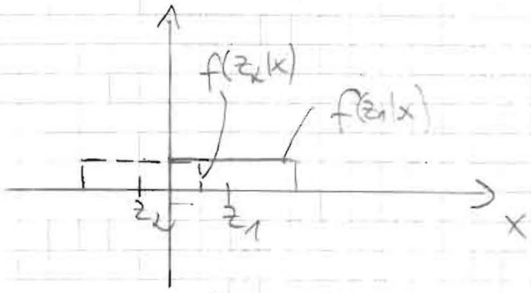
Consider different cases:

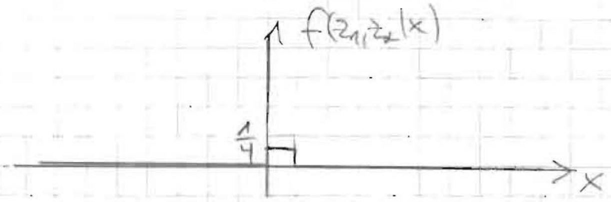
I) $|z_1 - z_2| > 2$



$f(z_1, z_2|x) = 0 \Rightarrow$ no value for x can explain z_1, z_2 given the model $z_i = x + w_i$

II) $|z_1 - z_2| \leq 2$





$$f(z_1, z_2|x) = \begin{cases} 1/4 & \text{for } x \in [z_1 - 1, z_1 + 1] \cap [z_2 - 1, z_2 + 1] \\ 0 & \text{otherwise} \end{cases}$$

intersection

$\Rightarrow \hat{x} \in [z_1 - 1, z_1 + 1] \cap [z_2 - 1, z_2 + 1]$ all equally optimal

Problem 3

Proceed similarly as in class: $z, w \in \mathbb{R}^m$, $x \in \mathbb{R}^n$

$$H = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_m \end{bmatrix} \qquad H_i = [\underbrace{h_{i1}, \ldots, h_{in}}_{\text{scalars}}]$$

$$f(z|x) \propto \exp\left(-\frac{1}{2}\left(\sum_{i=1}^{m} \frac{(z_i - H_i x)^2}{\sigma_i^2}\right)\right)$$

Differentiate w.r.t. x_j, set to 0:

$$\sum_{i=1}^{m} \frac{z_i - H_i x}{\sigma_i^2} h_{ij} = 0 \qquad j = 1, 2, \dots n$$

$$\underbrace{[h_{1j} \dots h_{mj}]}_{\text{column of } H} \underbrace{\begin{bmatrix} 1/\sigma_1^2 & & 0 \\ & 1/\sigma_2^2 & \\ 0 & & 1/\sigma_m^2 \end{bmatrix}}_{W} (z - Hx) = 0 \qquad j = 1, \dots n$$

$\Rightarrow H^T W (z - Hx) = 0$ with W weight matrix

$\Rightarrow \hat{x} = (H^T W H)^{-1} H^T W z$

Weighted least square:

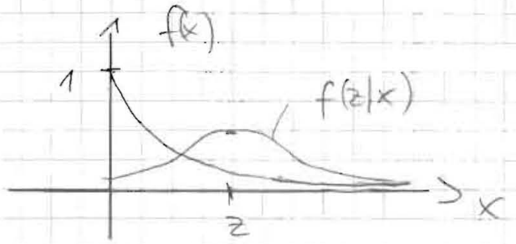
$w(x) = z - Hx$ $\qquad$ $x = \arg\min w(x)^T W w(x)$

weighting of measurements

Problem 4

exponential distribution:

$$f(x) = \begin{cases} 0 & x < 0 \\ \exp(-x) & x \geq 0 \end{cases}$$



$$f(z|x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(z-x)^2\right)$$

$$f(z|x)\, f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{\sqrt{2\pi}} \exp(-x) \exp\left(-\frac{1}{2}(z-x)^2\right) \quad (*) & x \geq 0 \end{cases}$$

Differentiate (*) w.r.t. x and set to 0:

$$-1 + (z - \hat{x}) \overset{!}{=} 0 \iff \underline{\underline{\hat{x} = z - 1}} \quad \text{if } z \geq 1$$

$$\Rightarrow \underline{\underline{\hat{x} = 0}} \quad \text{if } z < 1 \quad \text{(cf. figure)}$$

Problem 5

$A = [a_{ij}]$ $\quad i,j = 1,2$

element in ith row and jth column

$B = [b_{ij}]$ $\quad i,j = 1,2$, $b_{ij} = b_{ji}$

$$(AB)_{ij} = \sum_{r=1}^{2} a_{ir} b_{rj}$$

$$(ABA^T)_{ij} = \sum_{s=1}^{2} \left(\sum_{r=1}^{2} a_{ir} b_{rs} \right) a_{js}$$ from A^T

$$\mathrm{Tr}(ABA^T) = (ABA^T)_{11} + (ABA^T)_{22}$$
$$= \sum_{t=1}^{2} \sum_{s=1}^{2} \sum_{r=1}^{2} a_{tr} b_{rs} a_{ts}$$

$$\frac{\partial \mathrm{Tr}(ABA^T)}{\partial a_{ij}} = \sum_{s=1}^{2} \underbrace{b_{js}}_{= b_{sj}} a_{is} + \sum_{r=1}^{2} a_{ir} b_{rj},$$

$$= 2(AB)_{ij}$$

$$\Rightarrow \frac{\partial \mathrm{Tr}(ABA^T)}{\partial A} = 2AB \quad \text{q.e.d.}$$

Problem 6

Similar way of proceeding:

$$Tr(AB) = \sum_{s=1}^{2} \sum_{r=1}^{2} a_{\underset{ij}{sr}} b_{rs}$$

$$\frac{\partial\, Tr(AB)}{\partial a_{ij}} = b_{ji}$$

$$\Rightarrow \frac{\partial\, Tr(AB)}{\partial A} = B^T = [\, b_{ji} \,]$$

Problem 7

a) • $P(0) = 1$, $\hat{x}(0) = 0$,
• $H(k) = 1$, $R(k) = 1$

$$K(k) = \frac{P(k-1)}{P(k-1)+1} \quad (*)$$

$$P(k) = (1-K(k))^2 P(k-1) + K(k)^2$$
$$= \frac{P(k-1)}{(1+P(k-1))^2} + \frac{P(k-1)^2}{(1+P(k-1))^2}$$

$$= \frac{P(k-1)}{1+P(k-1)} \quad (**)$$

$$P(0) = 1 \quad P(1) = \frac{1}{2} \quad P(2) = \frac{1}{3} \quad \dots \quad P(k) = \frac{1}{1+k}$$

Proof by induction
→ see below

$$\Rightarrow P(k) = \frac{1}{1+k}, \quad K(k) = \frac{1}{1+k} \text{ from } (*)$$

Proof by induction:

- Assume $P(k) = \frac{1}{1+k}$ (1)
- Start with $P(0) = \frac{1}{1+0} = 1$ ✓
- Show $P(k+1) = \frac{1}{1+k+1} = \frac{1}{k+2} = \frac{P(k)}{1+P(k)}$ from $(\cancel{*})$

$$\underset{\substack{\uparrow \\ \text{use} \\ \text{Assumption (1)}}}{=} \frac{\frac{1}{1+k}}{1+\frac{1}{1+k}} = \frac{1}{k+2} \checkmark \qquad \text{q.e.d.}$$

b) see Code attached

→ recursive least squares works, needs around 3500 iterations / measurements to converge

→ recursive least squares is <u>optimal</u> estimator for Gaussian noise

c) see Code attached

→ recursive least squares works, but we can do much better with the proposed nonlinear estimator

d) The only possible combinations are

x	w	z
-1	-1	-2
-1	1	0
1	-1	0
1	1	2

$\hat{x}_{MMSE} = E[x|z]$

if $z=-2$ or $z=2$, we know x precisely: $x=-1$ or $x=1$, respectively; if $z=0$ it is equally likely that $x=1$ or $x=-1$

Note that we do not have continuous random variables; that is, we have to use the original definition

$$\hat{x}_{MMSE} := \underset{\hat{x}}{\arg\min} \; \underset{x}{E}\left[(\hat{x}-x)^2 \mid z\right] \quad \text{for scalar case}$$

- for $z = 2$, $\hat{x}_{MMSE} = 1$ since $f(x=-1 \mid z=2) = 0$
- for $z = -2$, $\hat{x}_{MMSE} = -1$ " $f(x=1 \mid z=-2) = 0$
- for $z = 0$, $\hat{x}_{MMSE} = \underset{\hat{x}}{\arg\min}\left(f(x=1 \mid z=0) \cdot (\hat{x}-1)^2 + f(x=-1 \mid z=0) \cdot (\hat{x}+1)^2 \right)$

$$= \underset{\hat{x}}{\arg\min}\left(\frac{1}{2}(\hat{x}-1)^2 + \frac{1}{2}(\hat{x}+1)^2 \right)$$

$$= \pm 1$$

```
% Problem Set3 - Problem 7
%
% *Recursive Least Squares*
%
% Recursive Filtering and Estimation
% Spring 2010
%
% --
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%
% --
% Revision history
% [01.04.10, AS]    first version
%


clear

% Use 0 for normally distributed, CRV, 1 for DRV equally likely at +-1
DISTRIBUTION_TYPE = 0;

% Set the state of the normally distributed random number generator.
randn('state',0);

% Number of time samples. 100, 1000, or 10000
N = 1e4;
xEstLin = zeros(N+1,1);
xEstNL = zeros(N+1,1);

% Generate the true value of x
if DISTRIBUTION_TYPE == 0
   xActual = randn;
else
   xActual = sign(randn);
end

% Simulation
xEstLin(1) = 0;
xEstNL(1) = 0;
for k = 1:N

    if DISTRIBUTION_TYPE == 0
        w = randn;
    else
        w = sign(randn);
    end

    % Generate simulated measurement
    z = xActual + w;

    % The optimal linear estimator
    K = 1/(1+k);
```

```
    xEstLin(k+1) = xEstLin(k) + K*(z - xEstLin(k));

    % The nonlinear estimator, which only makes sense for discrete random
    % variables. Case 1: we have already determined what x is
    if (abs(xEstNL(k)) > 0.5)
        xEstNL(k+1) = xEstNL(k);
    else
        % Otherwise, we have to look at the measurement.  If z = 0, then it
        % is equally likely that x = 1 or -1.  If z = 2, then x must be 1.
        % If z = -2, then x must be -1
        if (abs(z) < 1)
            xEstNL(k+1) = 0;
        elseif (z > 1)
            xEstNL(k+1) = 1;
        else
            xEstNL(k+1) = -1;
        end
    end
end

% Plot the results
figure(1)
plot(0:N,xEstLin,'*',0:N,xEstNL,'o',0:N,xActual,'r.');
legend('Linear Estimate','Non-linear Estimate','Actual');

findfigs
```