Eidgenössische Technische Hochschule Zürich
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Introduction to Recursive Filtering and Estimation

Spring 2010

Problem Set: Probability review

Notes:

- **Notation:** Unless otherwise noted, x, y, and z denote random variables, $f_x(x)$ (or the short hand $f(x)$) denotes the probability density function of x, and $f_{x|y}(x|y)$ (or $f(x|y)$) denotes the conditional probability density function of x conditioned on y. The expected value is denoted by $\mathrm{E}[\cdot]$, the variance is denoted by $\mathrm{Var}(\cdot)$ and $\Pr(Z)$ denotes the probability that the event Z occurs.
- Please report any error that you may find to the teaching assistants (`strimpe@ethz.ch` or `aschoellig@ethz.ch`).

Problem Set

Problem 1

Prove $f(x|y) = f(x) \Leftrightarrow f(y|x) = f(y)$.

Problem 2

Prove $f(x|y,z) = f(x|z) \Leftrightarrow f(x,y|z) = f(x|z)f(y|z)$.

Problem 3

Come up with an example where $f(x,y|z) = f(x|z)f(y|z)$ but $f(x,y) \neq f(x)f(y)$ for *continuous* random variables x, y, and z.

Problem 4

Come up with an example where $f(x,y|z) = f(x|z)f(y|z)$ but $f(x,y) \neq f(x)f(y)$ for *discrete* random variables x, y, and z.

Problem 5

Let x and y be binary random variables: $x, y \in \{0,1\}$. Prove that $f_{x,y}(x,y) \geq f_x(x) + f_y(y) - 1$, which is called *Bonferroni's inequality*.

Problem 6

Prove $\mathrm{E}[ax+b] = a\,\mathrm{E}[x] + b$, where a and b are constants.

Problem 7

Let x and y be scalar random variables. Prove that for any functions $g(\cdot)$ and $h(\cdot)$, $\mathrm{E}[g(x)h(y)] = \mathrm{E}[g(x)]\,\mathrm{E}[h(y)]$ if x and y are independent.

Problem 8

From above, it follows that if x and y are independent, then $\mathrm{E}[xy] = \mathrm{E}[x]\,\mathrm{E}[y]$. Is the converse true, that is, if $\mathrm{E}[xy] = \mathrm{E}[x]\,\mathrm{E}[y]$, does it imply that x and y are independent? If yes, prove it. If no, find a counter-example.

Problem 9

Let $x \in \mathcal{X}$ be a scalar valued continuous random variable and $g(x)$ a continuously differentiable, strictly monotonically increasing function. Prove the *law of the unconscious statistician*, that is, show that the expected value of $y = g(x)$ is given by

$$\mathrm{E}[y] = \int\limits_{x \in \mathcal{X}} g(x) f_x(x)\, dx.$$

Problem 10

Let y be a scalar continuous random variable with probability density function $f_y(y)$, and g be a function transforming y to a new variable x by $x = g(y)$. Prove the *change of variables formula*, i.e. the equation for the probability density function $f_x(x)$, under the assumption that g is continuously differentiable with $\frac{dg(y)}{dy} < 0 \; \forall y$.

Problem 11

Let $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ be independent, scalar valued discrete random variables. Let $z = x + y$. Prove that

$$f_z(z) = \sum_{y \in \mathcal{Y}} f_x(z - y) f_y(y) = \sum_{x \in \mathcal{X}} f_x(x) f_y(z - x),$$

thus the probability density function is the *convolution* of the individual probability density functions.

Problem 12

Let x be a scalar continuous random variable that takes on only non-negative values. Prove that, for $a > 0$,

$$\Pr(x \geq a) \leq \frac{\mathrm{E}[x]}{a},$$

which is called *Markov's inequality.*

Problem 13

Let x be a scalar continuous random variable. Let $\bar{x} = \mathrm{E}[x]$, $\sigma^2 = \mathrm{Var}(x)$. Prove that for any $k > 0$,

$$\Pr\left(|x - \bar{x}| \geq k\right) \leq \frac{\sigma^2}{k^2},$$

which is called *Chebyshev's inequality.*

Problem 14

Let $x_1, x_2, \ldots, x_n$ be independent random variables, each having a uniform distribution over $[0, 1]$. Let $m = \max\{x_1, x_2, \ldots, x_n\}$. Show that the cumulative distribution function of m, $F_m(\cdot)$, is given by

$$F_m(m) = m^n, \quad 0 \leq m \leq 1.$$

What is the probability density function of m?

Problem 15

Prove that $\mathrm{E}[x^2] \geq (\mathrm{E}[x])^2$. When do we have equality?

Problem 16

Suppose that x and y are independent scalar continuous random variables. Show that

$$\Pr(x \leq y) = \int\limits_{-\infty}^{\infty} F_x(y) f_y(y)\, dy.$$

Problem 17

Use Chebyshev's inequality to prove the *weak law of large numbers*. Namely, if $x_1, x_2, \dots$ are independent and identically distributed with mean μ and variance σ^2 then, for any $\varepsilon > 0$,

$$\Pr\left(\left|\frac{x_1 + x_2 + \cdots + x_n}{n} - \mu\right| > \varepsilon\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Problem 18

Suppose that x is a random variable with mean 10 and variance 15. What can we say about $\Pr(5 < x < 15)$?

Problem 19

Let x and y be independent random variables with means μ_x and μ_y and variances σ_x^2 and σ_y^2. Show that

$$\mathrm{Var}(xy) = \sigma_x^2 \sigma_y^2 + \mu_y^2 \sigma_x^2 + \mu_x^2 \sigma_y^2.$$

Problem 20

a) Use the method presented in class to obtain samples x of the *exponential distribution*

$$\hat{f}_x(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

from given samples u of a uniform distribution.

b) Exponential random variables are used to model failures of components that do not wear, for example solid state electronics. This is based on their property

$$\Pr(x > s + t | x > t) = \Pr(x > s) \;\; \forall s, t \geq 0.$$

A random variable with this property is said to be *memoryless*. Prove that a random variable with an exponential distribution satisfies this property.

Problem Set 1: Recursive Filtering & Estimation

Probability Review

Problem 1

- from definition of conditional probability, we have

$$f(x,y) = f(x|y)\,f(y) \quad \text{and} \quad f(x,y) = f(y|x)\,f(x)$$

and therefore $\quad f(x|y)\,f(y) = f(y|x)\,f(x) \qquad (*)$

- If $f(x|y) = f(x) \overset{(*)}{\Rightarrow} f(x)\,f(y) = f(y|x)\,f(x)$

$\Rightarrow f(y) = f(y|x)$

If $f(y|x) = f(y) \overset{(*)}{\Rightarrow} f(x|y)\,f(y) = f(y)\,f(x)$

$\Rightarrow f(x) = f(x|y)$

$\therefore \quad f(x|y) = f(x) \Leftrightarrow f(y|x) = f(y)$ q.e.d.

Problem 2

Follows directly from $f(x,y|z) = f(x|y,z)\,f(y|z)$

Problem 3

- Let $x, y, z \in [0,1]$ be continuous random variables.

- Consider joint pdf $f(x,y,z) = c \cdot g_x(x,z)\, g_y(y,z)$ where g_x and g_y are functions (not necessarily pdf's) and c is a constant such that

$$\int_0^1\int_0^1\int_0^1 f(x,y,z)\, dx\, dy\, dz = 1$$

Then

$$f(x,y|z) = \frac{f(x,y,z)}{f(z)} = \frac{c \cdot g_x(x,z)\, g_y(y,z)}{c \cdot G_x(z) \cdot G_y(z)},$$

with

$$G_x(z) = \int_0^1 g_x(x,z)\, dx, \quad G_y(z) = \int_0^1 g_y(y,z)\, dy$$

(since $f(z) = \int_0^1\int_0^1 c\, g_x(x,z)\, g_y(y,z)\, dx\, dy$

$$= c\int_0^1 G_x(z) \cdot g_y(y,z)\, dy = c\, G_x(z) \cdot G_y(z).)$$

Also

$$f(x|z) = \frac{f(x,z)}{f(z)} = \frac{\int_0^1 f(x,y,z)\, dy}{f(z)} = \frac{c \cdot g_x(x,z)\, G_y(z)}{c \cdot G_x(z)\, G_y(z)}$$

$$= \frac{g_x(x,z)}{G_x(z)}$$

$$f(y|z) = \frac{g_y(y,z)}{G_y(z)} \quad \text{(by analogy)}$$

Therefore, $f(x,y|z) = \dfrac{g_x(x,z) \cdot g_y(y,z)}{G_x(z)\, G_y(z)} = f(x|z) \cdot f(y|z)$.

- Now, let $g_x(x,z) = x + z$ and $g_y(y,z) = y + z$.

Then

$$f(x,y) = \int_0^1 f(x,y,z)\, dz = c\int_0^1 (x+z)(y+z)\, dz$$

$$= c\int_0^1 z^2 + (x+y)z + xy\, dz = c\left[\tfrac{1}{3}z^3 + \tfrac{1}{2}(x+y)z^2 + xyz\right]_{z=0}^{z=1}$$

$$= c\left(\tfrac{1}{3} + \tfrac{1}{2}(x+y) + xy\right)$$

$$f(x) = \int_0^1 f(x,y)\,dx = c\int_0^1 \tfrac{1}{3} + \tfrac{1}{2}x + \tfrac{1}{2}y + xy\,dx$$

$$= c\left(\tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{2}y + \tfrac{1}{2}y\right) = c\left(y + \tfrac{7}{12}\right)$$

$$f(y) = \int_0^1 f(x,y)\,dy = c\left(x + \tfrac{7}{12}\right) \qquad \text{(by symmetry)}$$

Therfore, $f(x)\,f(y) = c^2\left(\frac{49}{144} + \frac{7}{12}(x+y) + xy\right) \neq f(x,y)$.

Problem 4

- As in problem 3, let $f(x,y,z) = c\, g_x(x,z)\, g_y(y,z)$. This ensures $f(x,y\mid z) = f(x\mid z)\, f(y\mid z)$. Let $x, y, z \in \{0,1\}$

- Define g_x and g_y by value tables:

x	z	$g_x(x,z)$
0	0	0
0	1	1
1	0	1
1	1	0

y	z	$g_y(y,z)$
0	0	1
0	1	0
1	0	0
1	1	1

x	y	z	$f(x,y,z) = \frac{1}{2} g_x(x,z)\, g_y(y,z)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0.5
1	0	0	0.5
1	0	1	0
1	1	0	0
1	1	1	0

• Compute $f(x,y)$, $f(x)$, $f(y)$

x	y	$f(x,y)$
0	0	0
0	1	0.5
1	0	0.5
1	1	0

x	$f(x)$
0	0.5
1	0.5

y	$f(y)$
0	0.5
1	0.5

x	y	$f(x) \cdot f(y)$
0	0	0.25
0	1	0.25
1	0	0.25
1	1	0.25

$$\left(\begin{array}{l} f(x,y) = \sum_z f(x,y,z) = f(x,y,0) + f(x,y,1) \\ f(x) = f(x,0) + f(x,1) \end{array} \right)$$

$\therefore \quad f(x,y) \neq f(x)\, f(y)$

• As verification, compute $f(x,y|z)$, $f(x|z)$, $f(y|z)$, although by construction of $f(x,y,z)$ it is already clear that $f(x,y|z) = f(x|z)\, f(y|z)$.

x	y	z	$f(x,y\|z)$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	0	0
0	0	1	0
0	1	1	1
1	0	1	0
1	1	1	0

From $f(x,y,z)$ it follows that if $z=0$, then $(x,y)=(1,0)$ with probability 1

(or: compute from $f(x,y|z) = \dfrac{f(x,y,z)}{f(z)}$

similarly

x	z	$f(x\|z)$
0	0	0
1	0	1
0	1	1
1	1	0

y	z	$f(y\|z)$
0	0	1
1	0	0
0	1	0
1	1	1

x	y	z	$f(x\|z)\, f(y\|z)$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	0	0
0	0	1	0
0	1	1	1
1	0	1	0
1	1	1	0

Problem 5

- By marginalization,

$$f_X(x) = \sum_{y \in Y} f_{XY}(x,y) = f_{XY}(x,y) + f_{XY}(x,1-y)$$

$$f_Y(y) = \sum_{x \in X} f_{XY}(x,y) = f_{XY}(x,y) + f_{XY}(1-x,y)$$

- Therefore,

$$f_X(x) + f_Y(y) = 2 f_{XY}(x,y) + f_{XY}(x,1-y) + f_{XY}(1-x,y)$$

$$= \underbrace{f_{XY}(x,y) + f_{XY}(x,1-y) + f_{XY}(1-x,y) + f_{XY}(1-x,1-y)}_{= \sum_{y \in \{0,1\}} \sum_{x \in \{0,1\}} f_{XY}(x,y) = 1}$$

$$- f_{XY}(1-x,1-y) + f_{XY}(x,y)$$

$$= 1 + f_{XY}(x,y) - f_{XY}(1-x,1-y)$$

$$\leq 1 + f_{XY}(x,y)$$

$$\Rightarrow \quad f_{XY}(x,y) \geq f_X(x) + f_Y(y) - 1$$

q.e.d.

Problem 6

Using the law of the unconscious statistician, we have for a discrete random variable X

$$E[aX+b] = \sum_{x \in X} (ax+b) f_X(x)\, dx = a \underbrace{\sum_{x \in X} x f_X(x)}_{E[X]} + b \underbrace{\sum_{x \in X} f_X(x)}_{1}$$

$$= a\,E[X] + b$$

and for a continuous random variable X

$$E[aX+b] = \int_{-\infty}^{\infty} (ax+b) f_X(x)\, dx$$

$$= a \underbrace{\int_{-\infty}^{\infty} x f_X(x)\, dx}_{E[X]} + b \underbrace{\int_{-\infty}^{\infty} f_X(x)\, dx}_{1}$$

$$= a\,E[X] + b$$

q.e.d.

Problem 7

Using the law of the unconscious statistician applied on the vector random variable $\begin{bmatrix} X \\ Y \end{bmatrix}$ with pdf $f_{XY}(x,y)$, we have

$$E[g(x)h(y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)\, f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)\, f_X(x)\, f_Y(y)\, dx\, dy$$

(by independence x, y)

$$= \int_{-\infty}^{\infty} g(x) f_X(x)\, dx \cdot \int_{-\infty}^{\infty} h(y) \cdot f_Y(y)\, dy$$

$$= E[g(x)] \cdot E[h(y)].$$

and similarly for discrete random variables

Problem 8

We will construct a counterexample.

- Let x be a discrete r.v. that takes the values -1, 0, and 1 with probabilities $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$, respectively.
- Let y be a discrete r.v. taking the values -1, 1 with equal probability. Let x and y be independent.
- Let z be a discrete r.v. defined by $z = x \cdot y$.(*) By construction, it is clear that z depends on x. We will show that $E[xz] = E[x] \cdot E[z]$.

(*) It follows that z can take the values $-1, 0, 1$.

Pdf $f_{xz}(x,z)$:

x	z	$f_{xz}(x,z)$	
-1	-1	1/8	$\rightarrow$ $x=-1$ (Prob $\frac{1}{4}$) $\wedge$ $w=1$ (Prob $\frac{1}{2}$)
-1	0	0	$\rightarrow$ impossible since $w \neq 0$
-1	1	1/8	$\rightarrow$ $x=-1$ (Prob $\frac{1}{4}$) $\wedge$ $w=-1$ (Prob $\frac{1}{2}$)
0	-1	0	$\rightarrow$ impossible
0	0	1/2	$\rightarrow$ w either ± 1, $x=0$ (Prob. $\frac{1}{2}$)
0	1	0	⋮
1	-1	1/8	
1	0	0	
1	1	1/8	

Marginalization: $f_x(x)$, $f_z(z)$

x	$f(x)$
-1	1/4
0	1/2
1	1/4 ✓

z	$f(z)$
-1	1/4
0	1/2
1	1/4 ✓

It follows that $f_{xz}(x,z) = f_x(x)\, f_z(z) \ \forall x,z$ does not hold.
For example, $f_{xz}(-1,0) = 0 \neq 1/8 = f_x(-1)\, f_z(0)$

Expected values:

$$E[xz] = \sum_x \sum_z xz\, f_{xz}(x,z)$$
$$= (-1)(-1)\tfrac{1}{8} + (-1)(1)\tfrac{1}{8} + 0 \cdot 0 \cdot \tfrac{1}{2} + (1)(-1)\cdot\tfrac{1}{8} + (1)(1)\tfrac{1}{8}$$
$$= 0$$
$$E[x] = \sum_x x\, f_x(x)$$
$$= -\tfrac{1}{4} + \tfrac{1}{4} = 0$$
$$E[z] = 0$$

$$\therefore\ E[xz] = E[x]\,E[z] \ \not\Rightarrow\ f_{xz}(x,z) = f_x(x)\cdot f_z(z)$$

Problem 9

- From the change of variables formula (see problem 10), we have

$$f_Y(y) = \frac{f_X(x)}{\frac{dg}{dx}(x)} \quad \text{(since } g \text{ cont differentiable, strictly monotonic)}$$

- Furthermore, $y = g(x) \Rightarrow dy = \frac{dg}{dx}(x)\,dx$

- Let $\mathbb{Y} = g(\mathbb{X})$, that is $\mathbb{Y} = \{ y \mid \exists x \in \mathbb{X} : y = g(x) \}$.

- $$E[y] = \int_{y \in \mathbb{Y}} y\, f_Y(y)\, dy = \int_{x \in \mathbb{X}} g(x) \frac{f_X(x)}{\cancel{\frac{dg}{dx}(x)}} \cancel{\frac{dg}{dx}(x)}\, dx$$

$$= \int_{x \in \mathbb{X}} g(x)\, f_X(x)\, dx$$

q.e.d.

Problem 10

- Consider the probability that y is in a small interval $[\bar{y}, \bar{y} + \Delta y]$ (with $\Delta y > 0$ small),

$$P(y \in [\bar{y}, \bar{y} + \Delta y]) = \int_{\bar{y}}^{\bar{y}+\Delta y} f_Y(y)\, dy$$

which approaches $f_Y(\bar{y})\Delta y$ in the limit as $\Delta y \to 0$

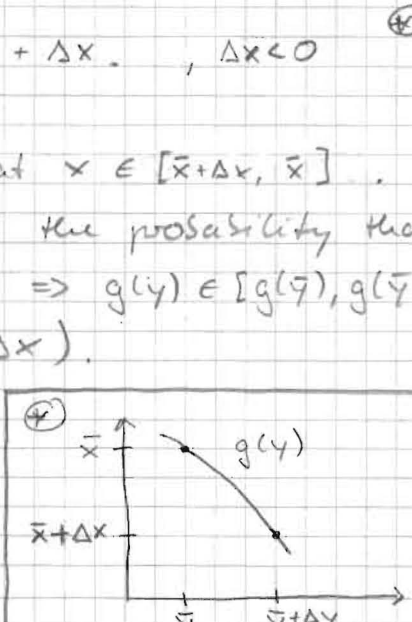
- Let $\bar{x} := g(\bar{y})$. For small Δy, we have (Taylor series)

$$g(\bar{y} + \Delta y) \approx g(\bar{y}) + \underbrace{\frac{dg}{dy}(\bar{y}) \cdot \Delta y}_{=: \Delta x} = \bar{x} + \Delta x. \quad , \Delta x < 0 \quad ⊛$$

- We are interested in the probability that $x \in [\bar{x} + \Delta x, \bar{x}]$.
For small Δx, this probability is equal the probability that $y \in [\bar{y}, \bar{y} + \Delta y]$ (since $y \in [\bar{y}, \bar{y} + \Delta y] \Rightarrow g(y) \in [g(\bar{y}), g(\bar{y} + \Delta y)]$ $\Rightarrow x \in [\bar{x} + \Delta x, \bar{x}]$ for small Δy and Δx).

$$\therefore \; P(y \in [\bar{y}, \bar{y} + \Delta y]) = f_Y(\bar{y})\Delta y$$
$$= P(x \in [\bar{x} + \Delta x, \bar{x}]) = -f_X(\bar{x})\Delta x$$

for small Δy (and thus small Δx)



g(y) decreasing ⇒ Δx neg.

$$\Rightarrow \quad f_Y(\bar{y})\Delta y = -f_X(\bar{x})\Delta x$$

$$\Rightarrow \quad f_X(\bar{x}) = \frac{f_Y(\bar{y})}{-\frac{\Delta x}{\Delta y}}$$

- As $\Delta y \to 0$ (and thus $\Delta x \to 0$), $\frac{\Delta x}{\Delta y} \to + \frac{dg}{dy}(\bar{y})$ and therefore

$$f_X(\bar{x}) = \frac{f_Y(\bar{y})}{-\frac{dg}{dy}(\bar{y})}$$

or

$$f_X(x) = \frac{f_Y(y)}{-\frac{dg}{dy}(y)} \quad .$$

Problem 11

- $f_Z(\bar{z}) = Pr\{z = \bar{z}\}$

The probability that z takes on $\bar{z}$ is given by the sum of the probabilities of all possible combinations of x and y such that $\bar{z} = x + y$, that is all $\bar{y} \in Y$ s.t. $x = \bar{z} - \bar{y}$ and $y = \bar{y}$ [or alternatively, $\bar{x} \in X$ s.t. $x = \bar{x}$ and $y = \bar{z} - \bar{x}$].

$$\therefore \quad f_Z(\bar{z}) = Pr\{z = \bar{z}\} = \sum_{\bar{y} \in Y} Pr\{x = \bar{z} - \bar{y}\} \cdot Pr\{y = \bar{y}\} \quad \text{(by independence assumption)}$$

$$= \sum_{\bar{y} \in Y} f_X(\bar{z} - \bar{y}) f_Y(\bar{y})$$

Rewrite $f_Z(z) = \sum_{y \in Y} f_X(z-y) f_Y(y)$

Similarly, $f_Z(z) = \sum_{x \in X} f_X(x) f_Y(z-x)$.

q.e.d.

Problem 12

$$E[x] = \int_0^\infty x\, f(x)\, dx = \underbrace{\int_0^a x\, f(x)\, dx}_{\geq 0} + \int_a^\infty x\, f(x)\, dx$$

$$\geq \int_a^\infty x\, f(x)\, dx \geq \int_a^\infty a\, f(x)\, dx$$

$$= a \int_a^\infty f(x)\, dx = a \Pr(x \geq a)$$

q.e.d.

Problem 13

Can be obtained as a corollary of Markov's inequality:

- $(x-\bar{x})^2$ is a nonnegative random variable; apply Markov's inequality with $a = k^2$ $(k>0)$:

$$\Pr\left((x-\bar{x})^2 \geq k^2\right) \leq \frac{E[(x-\bar{x})^2]}{k^2} = \frac{\mathrm{Var}(x)}{k^2} = \frac{\sigma^2}{k^2}$$

- Since $(x-\bar{x})^2 \geq k^2 \Leftrightarrow |x-\bar{x}| \geq k$,

$$\Pr(|x-\bar{x}| \geq k) \leq \frac{\sigma^2}{k^2}$$

q.e.d.

Problem 14

For $0 \leq t \leq 1$,

$$F_m(t) = \Pr(m \in [0,t]) = \Pr(\max\{x_1, \ldots, x_n\} \in [0,t])$$

$$= \Pr(x_i \in [0,t]\ \forall i \in \{1,\ldots,n\})$$

$$= \Pr(x_1 \in [0,t]) \cdot \Pr(x_2 \in [0,t]) \cdot \ldots \cdot \Pr(x_n \in [0,t])$$

$$= t^n \qquad (\text{with } t = m \to F_m(m) = m^n)$$

Since $F_m(m) = \int_0^m f_m(\tau)\, d\tau$,

$$f_m(m) = \frac{dF_m}{dm}(m) = \frac{d}{dm}(m^n) = n \cdot m^{n-1}.$$

Problem 15

Let $\bar{x} = E[x]$

$$Var(x) = \int_{-\infty}^{\infty} (x-\bar{x})^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 f(x)\,dx - 2\bar{x} \underbrace{\int_{-\infty}^{\infty} x f(x)\,dx}_{\bar{x}} + \bar{x}^2 \underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{1}$$

$$= E[x^2] - 2\bar{x}^2 + \bar{x}^2 = E[x^2] - \bar{x}^2$$

$$\therefore \quad E[x^2] = \bar{x}^2 + Var(x)$$
$$\geq \bar{x}^2 \qquad (\text{since } Var(x) \geq 0)$$

with equality if $Var(x) = 0$, i.e. when x is constant.

Problem 16

- For fixed $\bar{y} \in Y$,

  $Pr(x \leq \bar{y}) = F_x(\bar{y})$

- The probability that $y \in [\bar{y}, \bar{y}+\Delta y]$ is

  $Pr(y \in [\bar{y}, \bar{y}+\Delta y]) = \int_{\bar{y}}^{\bar{y}+\Delta y} f_y(y)\,dy \rightarrow f_y(\bar{y}) \cdot \Delta y$ for small Δy

- Therefore, the probability that $x \leq \bar{y}$ and $y \in [\bar{y}, \bar{y}+\Delta y]$ is

  $Pr(x \leq \bar{y} \wedge y \in [\bar{y}, \bar{y}+\Delta y]) = Pr(x \leq \bar{y}) \cdot Pr(y \in [\bar{y}, \bar{y}+\Delta y])$

  $= F_x(\bar{y}) \cdot f_y(\bar{y}) \Delta y \quad (*) \qquad$ (by indep. of x, y)

- Now, since we are interested in $Pr(x \leq y)$, i.e. in the probability of x being less than <u>any</u> y and not just a fixed $\bar{y}$, we can sum $(*)$ over all possible intervals $[\bar{y}_i, \bar{y}_i + \Delta y]$

  $$Pr(\exists \bar{y}_i \text{ such that } x \leq \bar{y}_i) = \sum_{i=-\infty}^{\infty} F_x(\bar{y}_i) \cdot f_y(\bar{y}_i) \cdot \Delta y$$

  By letting $\Delta y \rightarrow 0$, we obtain

  $$Pr(x \leq y) = \int_{-\infty}^{\infty} F_x(y)\, f_y(y)\,dy .$$

Problem 17

- Consider the random variable $\frac{X_1 + X_2 + \ldots + X_n}{n}$.

It has the mean

$$E\left[\frac{X_1 + X_2 + \ldots + X_n}{n}\right] = \frac{1}{n}\left(E[X_1] + \ldots + E[X_n]\right) = \frac{1}{n}(n \cdot \mu) = \mu$$

and variance

$$\operatorname{Var}\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right) = E\left[\left(\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right)^2\right]$$

$$= \frac{1}{n^2}\left(E\left[(X_1-\mu)^2 + \ldots + (X_n-\mu)^2 + \text{"cross terms"}\right]\right)$$

where the expected value of the cross terms is zero, for example,

$$E[(X_i-\mu)(X_j-\mu)] = E[X_i-\mu] \cdot E[X_j-\mu] \text{ for } i \neq j \text{, by independence}$$
$$= 0$$

$$\therefore \operatorname{Var}\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right) = \frac{1}{n^2}\left(\operatorname{Var}(X_1) + \ldots + \operatorname{Var}(X_n)\right) = \frac{n \cdot \sigma^2}{n^2} = \frac{\sigma^2}{n}$$

- Applying Chebyshev's inequality with $\epsilon > 0$:

$$\Pr\left(\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| \geq \epsilon\right) \leq \frac{\sigma^2}{n \cdot \epsilon^2} \rightarrow 0 \text{ as } n \to \infty$$

$$\Rightarrow \Pr\left(\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| > \epsilon\right) \to 0 \text{ as } n \to \infty$$

Note: $\Pr\left(\left|\frac{X_1 + \ldots + X_n}{n} - \mu\right| \geq \epsilon\right) =$

$$\Pr\left(\left|\frac{X_1 + \ldots + X_n}{n} - \mu\right| > \epsilon\right) + \underbrace{\Pr\left(\left|\frac{X_1 + \ldots + X_n}{n} - \mu\right| = \epsilon\right)}_{=0 \text{ since } \frac{X_1 + \ldots + X_n}{n} \text{ is a continuous random variable}}$$

Problem 18

- $\Pr(5 < x < 15)$
  $= \Pr(|x-10| < 5) = 1 - \Pr(|x-10| \geq 5)$

- Chebyshev's inequality with $\bar{x} = 10$, $\sigma^2 = 15$, $k = 5$
  $\Pr(|x-10| \geq 5) \leq \frac{15}{5^2} = \frac{15}{25} = \frac{3}{5}$

$\therefore \quad \Pr(5 < x < 15) \geq 1 - \frac{3}{5} = \frac{2}{5}$.

Problem 19

- Mean: $E[xy] = E[x] \cdot E[y] = \mu_x \mu_y$

- $\mathrm{Var}(xy) = E[(xy)^2] - (\mu_x \mu_y)^2$
  $= E[x^2]\, E[y^2] - \mu_x^2 \mu_y^2$

- Using $E[x^2] = \sigma_x^2 + \mu_x^2$ and $E[y^2] = \sigma_y^2 + \mu_y^2$, we have

  $\mathrm{Var}(xy) = (\sigma_x^2 + \mu_x^2)(\sigma_y^2 + \mu_y^2) - \mu_x^2 \mu_y^2$
  $= \sigma_x^2 \sigma_y^2 + \mu_y^2 \sigma_x^2 + \mu_x^2 \sigma_y^2$

q.e.d.

Problem 20

a) • cummulative distribution:

$$\hat{F}_X(x) = \int_0^x \lambda e^{-\lambda t}\, dt = \left[-e^{-\lambda t}\right]_{t=0}^{x} = 1 - e^{-\lambda x}$$

• u: sample from uniform distribution. According to method presented in class, obtain sample x by solving $u = \hat{F}_X(x)$ for x:

$$u = 1 - e^{-\lambda x} \quad \Leftrightarrow \quad e^{-\lambda x} = 1-u$$

$$\Leftrightarrow \quad -\lambda x = \ln(1-u)$$

$$\Leftrightarrow \quad x = \frac{-\ln(1-u)}{\lambda}$$

• $u \to 0 \Rightarrow x \to 0$,

$u \to 1 \Rightarrow x \to +\infty$

b)

$$Pr(x > s+t \mid x > t)$$

$$= \frac{Pr(x > s+t \wedge x > t)}{Pr(x>t)} = \frac{Pr(x > s+t)}{Pr(x > t)}$$

$$= \frac{\int_{s+t}^{\infty} \lambda e^{-\lambda x}\, dx}{\int_{t}^{\infty} \lambda e^{-\lambda x}\, dx} = \frac{\left[-e^{-\lambda x}\right]_{s+t}^{\infty}}{\left[-e^{-\lambda x}\right]_{t}^{\infty}} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}}$$

$$= e^{-\lambda s} = \int_s^{\infty} \lambda e^{-\lambda x}\, dx$$

$$= Pr(x > s)$$

q.e.d.