ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

IMRT Measurement and Control Laboratory
Prof. R. D'Andrea
Prof. L. Guzzella

151-0563-00 **Dynamic Programming and Optimal Control** (Fall 2008)

Problem Set #4 Topic: Problems with Perfect State Information

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Angela Schöllig (aschoellig@ethz.ch) , November 25, 2008

"Design a Linear Quadratic Regulator (LQR) for the Sideways Motion of a Quadrocopter!"

For our quadrocopter, see Figure 1(a), which is currently executing first maneuvers in the 'Flying Machine Arena' in ML hall, a controller is to be designed with the goal of performing fast sideways motions.

(a) Picture of the real quadrocopter. (b) Schematic of the 2D model.

Figure 1: The Quadrocopter.

The controller design is based on a 2D model of the quadrocopter illustrated in Figure 1(b):

$$\begin{aligned}\ddot{y}(t) &= -a(t)\sin\left(\theta\left(t\right)\right)\\ \ddot{z}(t) &= a(t)\cos\left(\theta\left(t\right)\right) - g\\ \ddot{\theta}(t) &= q(t),\end{aligned}$$

where $a(t)$ and $u(t)$ represent the control inputs to the system. The gravitational constant is approximated by $[10\,\mathrm{m/s^2}]$. The position variables $y(t)$ and $z(t)$ have units of [m], θ is given in [rad], and the inputs $a(t)$ and $u(t)$ are in $[\mathrm{m/s^2}]$ and $[\mathrm{rad/s^2}]$, respectively.

Only concentrating on the horizontal control, the input $a(t)$ is set to

$$a(t) = \frac{10}{\cos\left(\theta(t)\right)},$$

resulting in $\ddot{z}(t) = 0$ and the simplified dynamics

$$\ddot{y}(t) = -10\tan\left(\theta\left(t\right)\right) \quad (1)$$

$$\ddot{\theta}(t) = q(t). \quad (2)$$

1. Linearize Equations (1),(2) about $\theta = 0$.

   We will control the system with a digital computer. Let τ be the sampling period and define time-discrete states as follows

   $$\begin{aligned} x_1(k) &= y(k\tau) \\ x_2(k) &= \dot{y}(k\tau) \\ x_3(k) &= \theta(k\tau) \\ x_4(k) &= \dot{\theta}(k\tau), \qquad k = 0,\, 1,\, 2,\, \ldots \end{aligned}$$

   Find an linear, time-discrete expression

   $$x(k+1) = Ax(k) + Bu(k),$$

   of the quadrocopter dynamics with $x(k) = [x_1(k),\, x_2(k),\, x_3(k),\, x_4(k)]^T$ and $q(t) = u(k)$ for $k\tau \leq t \leq (k+1)\tau$.

2. Our objective is to design an *infinite horizon LQR controller* that brings the system from the initial state,

   $$y(0) = 1, \quad \dot{y}(0) = \theta(0) = \dot{\theta}(0) = 0,$$

   to the final state,

   $$y(T) = \dot{y}(T) = \theta(T) = \dot{\theta}(t) = 0,$$

   for a value T *as small as possible.*

   In particular, we want to find a gain matrix F, such that, for $u(k) = Fx(k)$ and the initial condition $x(0) = [1,\, 0,\, 0,\, 0]^T$,

   $$x(k) \;\rightarrow\; 0 \qquad \text{for} \;\; k \;\rightarrow\; \infty.$$

   In addition, we have constraints on the input $u(k)$,

   $$|u(k)| = |q(k\tau)| \leq 100, \qquad \forall\, k,$$

   since the vehicle is limited in how quickly it can rotate. Furthermore, the angle $x_3(k)$ is constrained by

   $$|x_3(k)| = |\theta(k\tau)| \leq \frac{\pi}{6}, \qquad \forall\, k,$$

   guaranteeing that the linearization is reasonably accurate and also that $a(t) = 10/\left(\cos\left(\theta(t)\right)\right)$ is feasible. Finally, our sampling period is $\tau = 1/50$.

   By appropriately choosing the matrices Q and R and using the `dare` function in MATLAB, find a feedback control strategy $u(k) = Fx(k)$, which brings the system to within

   $$|x_i(k)| \leq 0.01, \qquad i = 1,\, 2,\, 3,\, 4, \tag{3}$$

   as quickly as possible while satisfying the constraints.[1]

   This will be an iterative process and numerical in nature. In particular, there is no direct way to capture the constraints in the LQR design or to minimize the time, it takes to get within a tolerance of the destination.

[1] Note that the time T, to be minimized, corresponds to the time, at which conditions (3) are fulfilled for the first time.

You will have to find the solution iteratively by modifying the matrices Q and R based on your simulation results.

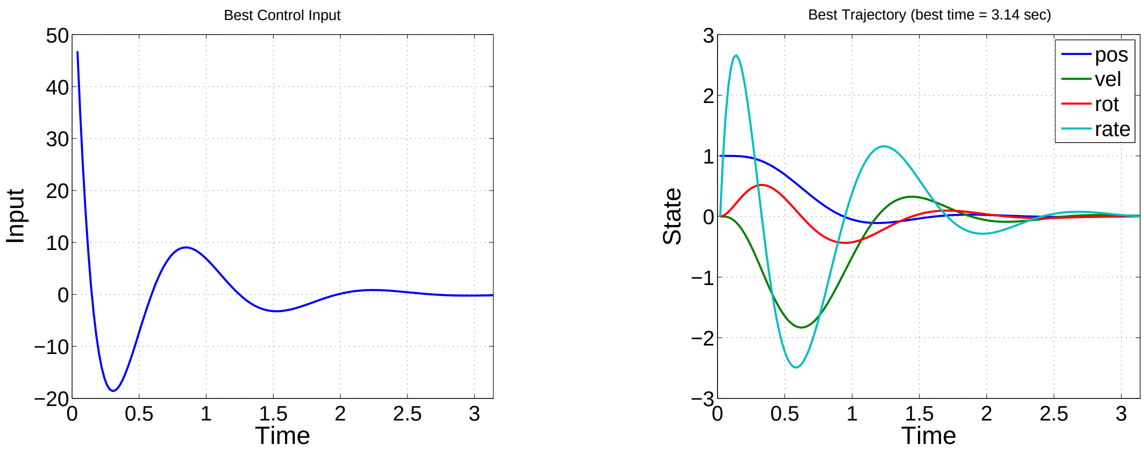
Please hand in a short description of your solution strategy, your well-commented Matlab *code, and, of course, the set of best parameters Q, R and the resulting F and T. Include plots showing the performance of your quadrocopter.*

3. Using the results from Exercise 2 as a starting point, develop, how much you can improve your design by using a *finite horizon LQR controller*.

   Show your improvements by plots and explain how you got your solution. Also attach your Matlab *code. What is your best choice for Q_k, R_k and your minimum time T?*

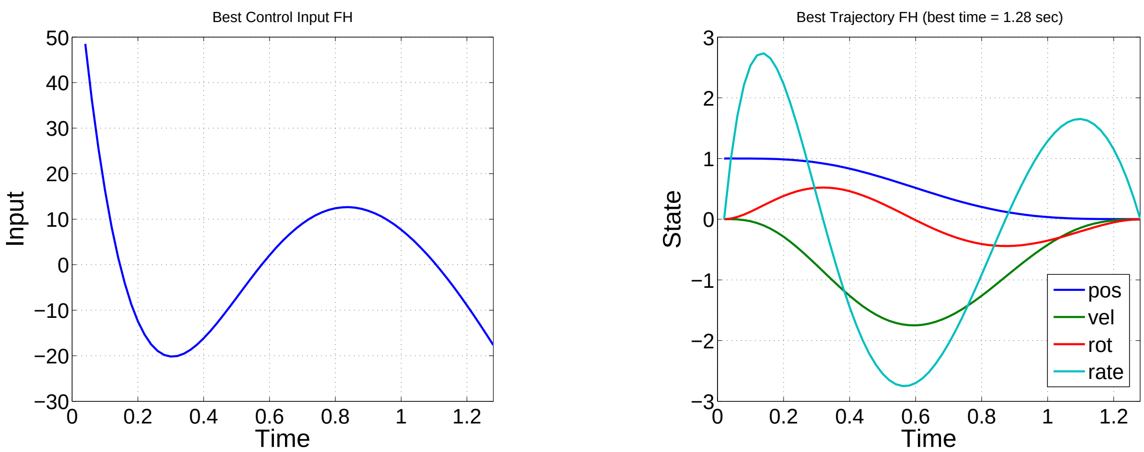
"Who can do best?"

And this is what Prof. D'Andrea got:



(a) The input. (b) The state trajectories.

Figure 2: Results for the infinite horizon LQR controller.



(a) The input. (b) The state trajectories.

Figure 3: Results for the finite horizon LQR controller.

4. BERTSEKAS, p. 211, exercise 4.22

5. BERTSEKAS, p. 212, exercise 4.23

Exercises 4 to 5 are taken from the book *Dynamic Programming and Optimal Control by Dimitri P. Bertsekas, Vol. I, 3rd edition, 2005, 558 pages, hardcover.*