ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

IMRT Measurement and Control Laboratory
Prof. R. D'Andrea
Prof. L. Guzzella

151-0563-00 **Dynamic Programming and Optimal Control** (Fall 2008)

Problem Set #2 Topic: Static Optimization, Markov Chains, Shortest Path Problems

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Angela Schöllig (aschoellig@ethz.ch) , 7. Oktober 2008

1. Find the maxima of the function

$$f(x) = \exp\left(-x_1^2 - x_2^2\right) \qquad \text{subject to} \quad h(x) = x_2 + x_1^2 - 1 = 0\,,$$

where $x = [x_1, x_2]^T \in \mathbb{R}^2$.

a) First, check if the problem can be formulated as a convex optimization problem. Plot the function $f(x)$ for feasible values $\left\{x \in \mathbb{R}^2 \,|\, h(x) = 0\right\}$ using the MATLAB plot function `plot3`. Does a unique maximum exist? Submit your MATLAB code together with the corresponding 3D plot for $x_1 \in [-2, 2]$.

b) Solve the problem analytically. Calculate the Lagrangian and state the first order optimality conditions for this problem. Which points $x \in \mathbb{R}^2$ satisfy the optimality conditions? Characterize these points.

c) Solve the problem numerically. Familiarize yourself with the MATLAB function `fmincon` and try different initial guesses $x_{0,1} = [0, 0]^T$, $x_{0,2} = [-1, -1]^T$, $x_{0,3} = [1, 1]^T$. What happens if you choose $x_{0,4} = [100, 1000]^T$? Submit both, your MATLAB code and a plot showing the calculated extrema on the curve generated in part 1.a).

2. Consider a gambler who at each play of the game has probability p of winning a dollar and probability $q = 1 - p$ of losing a dollar. Assuming that successive plays of the game are independent, what is the probability that, starting with i dollar, the gambler's fortune will reach his final goal, namely N dollar, before going broke and not being able to gamble anymore, i.e., before reaching 0 dollar?

a) Model the problem as a finite-state Markov chain. Define an appropriate set of states and draw a state transition diagram.

b) Classify the states. Do recurrent classes of states exist? Interpret your findings.

c) Derive an expression for the probability p_i, $i = 0, 1, 2, \ldots, N$, denoting the probability that, starting with i dollar, the gambler's fortune will eventually reach his goal of N dollar. What happens for $N \to \infty$?

3. The weather in some area is classified as either "sunny", "cloudy", or "rainy" on a given day. Let x_k denote the weather state on the kth day, $k = 1, 2, \ldots$, and map the states "sunny", "cloudy", and "rainy" into the numbers 0, 1, 2, respectively. Suppose transition probabilities from one day to the next are summarized through the matrix:

$$P = \begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.5 & 0.3 & 0.2 \\ 0.1 & 0.5 & 0.4 \end{bmatrix}.$$

a) Draw a state transition diagram for this chain.

b) Assuming the weather is cloudy today, predict the weather for the next two days. What is the probability of having sunny weather six days later?

c) Find the stationary state probabilities (if they exist) for this chain. If they do not exist, explain why this is so.

4. BERTSEKAS, p. 98, exercise 2.1
5. BERTSEKAS, p. 98, exercise 2.2

   Note that the figure at the top of p. 99 should be named Figure 2.4.2, not Figure 2.4.1.
6. BERTSEKAS, p. 103, exercise 2.14

Exercises 4 to 6 are taken from the book *Dynamic Programming and Optimal Control by Dimitri P. Bertsekas, Vol. I, 3rd edition, 2005, 558 pages, hardcover.*