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Midterm Examination **November 12th, 2008**

Dynamic Programming & Optimal Control (151-0563-00) Prof. R. D'Andrea

Solutions

Exam Duration:	**150 minutes**
Number of Problems:	**4** (25% each)
Permitted aids:	Textbook *Dynamic Programming and Optimal Control by Dimitri P. Bertsekas, Vol. I, 3rd edition, 2005, 558 pages.* Your written notes. **No** calculators.
Important:	Use only these prepared sheets for your solutions.

Problem 1 25%

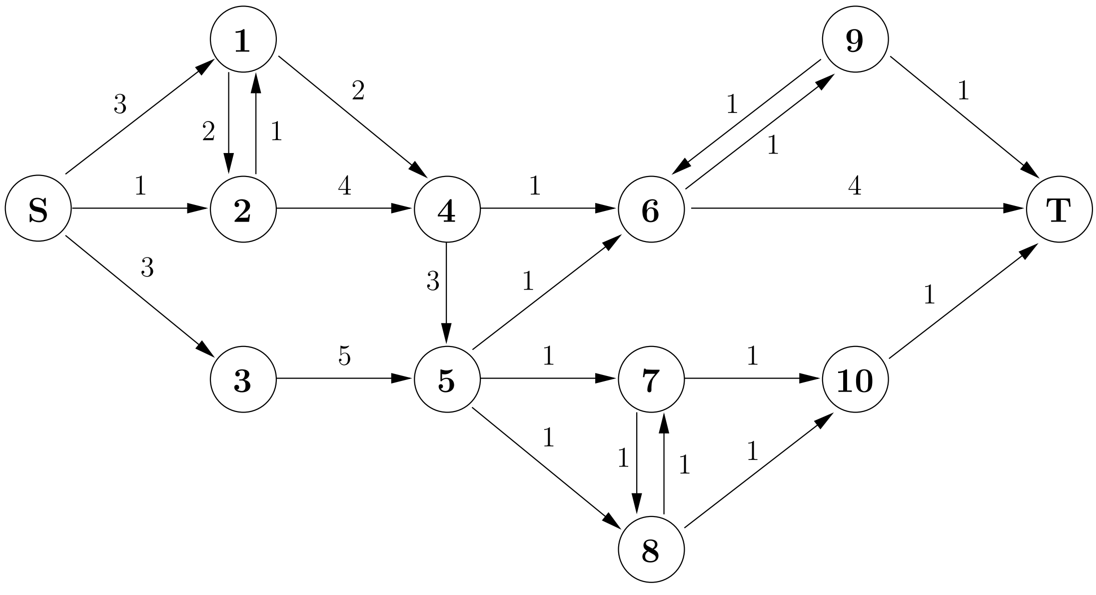

Figure 1

Find the shortest path from node S to node T for the graph given in Figure 1. Apply the *label correcting method*. Use *best-first search* to determine at each iteration which node to remove from `OPEN`; that is, remove node i with

$$d_i = \min_{j \text{ in } \texttt{OPEN}} d_j,$$

where the variable d_i denotes the length of the shortest path from node S to node i that has been found so far.

Solve the problem by populating a table of the following form:

Iteration	Node exiting `OPEN`	`OPEN`	d_S	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	$d_T =$ `UPPER`
...														

Solution 1

Iteration	Node exiting `OPEN`	`OPEN`	d_S	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	$d_T =$ `UPPER`
0	-	S	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	S	1,2,3	0	3	1	3	∞	∞	∞	∞	∞	∞	∞	∞
2	2	1,3,4	0	2	1	3	5	∞	∞	∞	∞	∞	∞	∞
3	1	3,4	0	2	1	3	4	∞	∞	∞	∞	∞	∞	∞
4	3	4,5	0	2	1	3	4	8	∞	∞	∞	∞	∞	∞
5	4	5,6	0	2	1	3	4	7	5	∞	∞	∞	∞	∞
6	6	5,9	0	2	1	3	4	7	5	∞	∞	6	∞	9
7	9	5	0	2	1	3	4	7	5	∞	∞	6	∞	7
8	5	-	0	2	1	3	4	7	5	∞	∞	6	∞	7

The shortest path is $S \rightarrow 2 \rightarrow 1 \rightarrow 4 \rightarrow 6 \rightarrow 9 \rightarrow T$ with a total length of 7.

Problem 2 **25%**

Consider the dynamic system

$$x_{k+1} = (1-a)\, w_k + a u_k, \qquad 0 \le a \le 1, \qquad k = 0, 1\,,$$

with initial state $x_0 = -1$. The cost function, to be minimized, is given by

$$\underset{w_0, w_1}{E} \left\{ x_2^2 \;+\; \sum_{k=0}^{1} \left(x_k^2 + u_k^2 + w_k^2 \right) \right\}.$$

The disturbance w_k takes the values 0 and 1. If $x_k \ge 0$, both values have equal probability. If $x_k < 0$, the disturbance w_k is 0 with probability 1. The control u_k is constrained by

$$0 \le u_k \le 1, \qquad k = 0, 1.$$

Apply the Dynamic Programming algorithm to find the optimal control policy and the optimal final cost $J_0(-1)$.

Solution 2

The optimal control problem is considered over a time horizon $N = 2$ and the cost, to be minimized, is defined by

$$g_2(x_2) = x_2^2 \qquad \text{and} \qquad g_k(x_k, u_k, w_k) = x_k^2 + u_k^2 + w_k^2, \quad k = 0, 1.$$

The DP algorithm proceeds as follows:

2nd stage:

$$J_2(x_2) = x_2^2$$

1st stage:

$$\begin{aligned} J_1(x_1) &= \min_{0 \le u_1 \le 1} E\left\{x_1^2 + u_1^2 + w_1^2 + J_2(x_2)\right\} \\ &= \min_{0 \le u_1 \le 1} \underset{w_1}{E}\left\{x_1^2 + u_1^2 + w_1^2 + J_2\big((1-a)\,w_1 + au_1\big)\right\} \\ &= \min_{0 \le u_1 \le 1} \underset{w_1}{E}\left\{x_1^2 + u_1^2 + w_1^2 + \big((1-a)\,w_1 + au_1\big)^2\right\} \end{aligned}$$

Distinguish two cases: $x_1 \ge 0$ and $x_1 < 0$.

I) $x_1 \ge 0$:

$$J_1(x_1) = \min_{0 \le u_1 \le 1} \underbrace{\left\{x_1^2 + u_1^2 + \frac{1}{2}\left(1 + ((1-a) + au_1)^2\right) + \frac{1}{2}\left(0 + ((1-a)\cdot 0 + au_1)^2\right)\right\}}_{L(x_1,u_1)}$$

Find the minimizing $\bar{u}_1$ by

$$\left.\frac{\partial L}{\partial u_1}\right|_{\bar{u}_1} = (1-a)\,a + 2\left(1+a^2\right)\bar{u}_1 \overset{!}{=} 0 \quad \Leftrightarrow \quad \bar{u}_1 = \frac{-a\,(1-a)}{2\,(1+a^2)} \le 0 \ \ (!).$$

Recall that the feasible set of inputs u_1 is given by $0 \le u_1 \le 1$.

However, using the information that $L\,(x_1, u_1)$ is convex in u_1; that is,

$$\frac{\partial^2 L}{\partial u_1^2} = 2\left(1+a^2\right) > 0,$$

it follows that $\bar{u}_1$ is a local minimum and the feasible optimal control u_1^* is given by

$$\Rightarrow \ u_1^* = \mu_1^*\,(x_1) = 0 \qquad \forall\, x_1 \ge 0.$$

II) $x_1 < 0$:

$$J_1(x_1) = \min_{0 \le u_1 \le 1} \underbrace{\left\{x_1^2 + \left(1 + a^2\right) u_1^2\right\}}_{L(x_1, u_1)}$$

Find the minimizing $\bar{u}_1$ by

$$\left.\frac{\partial L}{\partial u_1}\right|_{\bar{u}_1} = 2\left(1 + a^2\right)\bar{u}_1 \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \bar{u}_1 = 0.$$

Since the sufficient condition for a local minimum, $\left.\frac{\partial^2 L}{\partial u_1^2}\right|_{\bar{u}_1} > 0$, holds, the optimal control is

$$\Rightarrow \; u_1^* = \mu_1^*(x_1) = 0 \qquad \forall\, x_1 < 0.$$

0th stage:

$$J_0(-1) = \min_{0 \le u_0 \le 1} \mathop{E}_{w_0}\left\{(-1)^2 + u_0^2 + w_0^2 + J_1\big((1-a)\,w_0 + a u_0\big)\right\}$$

Since $x_0 < 0$, we get

$$J_0(-1) = \min_{0 \le u_0 \le 1} \underbrace{\left\{1 + u_0^2 + J_1\big(a u_0\big)\right\}}_{L(x_0, u_0)},$$

where $a u_0 \ge 0$. From above's results, the optimal cost-to-go function for $x_1 \ge 0$ is

$$J_1(x_1) = \frac{1}{2} + \frac{1}{2}(1-a)^2 + x_1^2.$$

Finally, the minimizing $\bar{u}_0$ results from

$$\left.\frac{\partial L}{\partial u_0}\right|_{\bar{u}_0} = 2\bar{u}_0 + 2a^2\bar{u}_0 \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \bar{u}_0 = 0.$$

Since $\left.\frac{\partial^2 L}{\partial u_0^2}\right|_{\bar{u}_0} > 0$, the optimal control u_0^* is

$$\Rightarrow \; u_0^* = \mu_0^*(-1) = 0.$$

With this, the optimal final cost reads as

$$J_0(-1) = \frac{3}{2} + \frac{1}{2}(1-a)^2.$$

In brief, the optimal control policy is to always set the input to zero, which can also be verified by carefully looking at the equations given in the problem statement.

Problem 3 25%

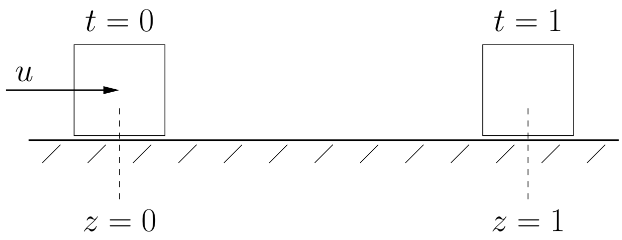

Figure 2

At time $t = 0$, a unit mass is at rest at location $z = 0$. The mass is on a frictionless surface and it is desired to apply a force $u(t)$, $0 \leq t \leq 1$, such that at time $t = 1$, the mass is at location $z = 1$ and again at rest. In particular,

$$\ddot{z}(t) = u(t), \quad 0 \leq t \leq 1, \tag{1}$$

with initial and terminal conditions:

$$z(0) = 0, \quad \dot{z}(0) = 0,$$
$$z(1) = 1, \quad \dot{z}(1) = 0.$$

Of all the functions $u(t)$ that achieve the above objective, find the one that minimizes

$$\frac{1}{2}\int_0^1 u^2(t)dt.$$

Hint: The state for this system is $x(t) = [\,x_1(t),\, x_2(t)\,]^T$, where $x_1(t) = z(t)$ and $x_2(t) = \dot{z}(t)$.

Solution 3

Introduce the state vector

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} z \\ \dot{z} \end{bmatrix}.$$

Using this notation, the dynamics read as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix}$$

with initial and terminal conditions,

$$\begin{aligned} x_1(0) &= 0, \quad x_2(0) = 0, \\ x_1(1) &= 1, \quad x_2(1) = 0. \end{aligned}$$

Apply the Minimum Principle.

- The Hamiltonian is given by

$$\begin{aligned} H(x,u,p) &= g(x,u) + p^T f(x,u) \\ &= \frac{1}{2}u^2 + p_1 x_2 + p_2 u. \end{aligned}$$

- The optimal input $u^*(t)$ is obtained by minimizing the Hamiltonian along the optimal trajectory. Differentiating the Hamiltonian with respect to u yields,

$$u^*(t) + p_2(t) = 0 \quad \Leftrightarrow \quad u^*(t) = -p_2(t).$$

  Since the second derivative of H with respect to u is 1, $u^*(t)$ is indeed a minimum.

- The adjoint equations,

$$\begin{aligned} \dot{p}_1(t) &= 0 \\ \dot{p}_2(t) &= -p_1(t), \end{aligned}$$

  are integrated and result in the following equations:

$$\begin{aligned} p_1(t) &= c_1, \quad c_1 \text{ constant} \\ p_2(t) &= -c_1 t - c_2, \quad c_2 \text{ constant.} \end{aligned}$$

  Using this result, the optimal input is given by

$$u^*(t) = c_1 t + c_2.$$

- Recalling the initial and terminal conditions on x, we can solve for c_1 and c_2.
  With above's results, the optimal state trajectory $x_2^*(t)$ is

$$\dot{x}_2^*(t) = c_1 t + c_2 \quad \Rightarrow \quad x_2^*(t) = \frac{1}{2}c_1 t^2 + c_2 t + c_3, \quad c_3 \text{ constant,}$$

and, therefore,

$$x_2^*(0) = 0 \quad \Rightarrow \quad c_3 = 0$$
$$x_2^*(1) = 0 \quad \Rightarrow \quad \frac{1}{2}c_1 + c_2 = 0 \quad \Rightarrow \quad c_1 = -2c_2,$$

yielding to

$$x_2^*(t) = -c_2t^2 + c_2t.$$

The optimal state $x_1^*(t)$ is given by

$$\dot{x}_1^*(t) = x_2^*(t) = -c_2t^2 + c_2t \quad \Rightarrow \quad x_1^*(t) = -\frac{1}{3}c_2t^3 + \frac{1}{2}c_2t^2 + c_4, \quad c_4 \text{ constant.}$$

With the conditions on x_1, we get

$$x_1^*(0) = 0 \quad \Rightarrow \quad c_4 = 0$$
$$x_1^*(1) = 1 \quad \Rightarrow \quad -\frac{1}{3}c_2 + \frac{1}{2}c_2 = 1 \quad \Rightarrow \quad c_2 = 6 \quad \text{and} \quad c_1 = -12.$$

- Finally, we obtain the optimal control

$$u^*(t) = -12t + 6,$$

and the optimal state trajectory

$$x_1^*(t) = z^*(t) = -2t^3 + 3t^2$$
$$x_2^*(t) = \dot{z}^*(t) = -6t^2 + 6t.$$

Problem 4 **25%**

Recall the Minimum Principle.

Under suitable technical assumptions, the following Proposition holds:
Given the dynamic system

$$\dot{x} = f\left(x(t),\, u(t)\right), \quad x(0) = x_0, \quad 0 \leq t \leq T$$

and the cost function,

$$h\left(x(T)\right) + \int_0^T g\left(x(t),\, u(t)\right)\, dt,$$

to be minimized, define the Hamiltonian function

$$H(x,u,p) = g\left(x,u\right) + p^T f\left(x,u\right).$$

Let $u^*(t),\ t \in [0,T]$ be an optimal control trajectory and $x^*(t)$ the resulting state trajectory. Then,

1. $\dot{p}(t) = -\frac{\partial H}{\partial x}\left(x^*(t), u^*(t), p(t)\right), \qquad p(T) = \frac{\partial h}{\partial x}\left(x^*(T)\right),$
2. $u^*(t) = \arg\min_{u \in U} H\left(x^*(t), u, p(t)\right),$
3. $H\left(x^*(t), u^*(t), p(t)\right)$ is constant.

Show that if the dynamics and the cost are time varying – that is, $f\left(x,u\right)$ is replaced by $f\left(x,u,t\right)$ and $g\left(x,u\right)$ is replaced by $g\left(x,u,t\right)$ – the Minimum Principle becomes:

1. $\dot{p}(t) = -\frac{\partial H}{\partial x}\left(x^*(t), u^*(t), p(t), t\right), \quad p(T) = \frac{\partial h}{\partial x}\left(x^*(T)\right),$
2. $u^*(t) = \arg\min_{u \in U} H\left(x^*(t), u, p(t), t\right)$
3. $H\left(x^*(t), u^*(t), p(t), t\right)$ not necessarily constant,

where the Hamiltonian function is now given by

$$H(x,u,p,t) = g\left(x,u,t\right) + p^T f\left(x,u,t\right).$$

Solution 4

General idea:

Convert the problem to a time-independent one, apply the standard Minimum Principle presented in class, and simplify the obtained equations.

Follow the subsequent steps:

- Introduce an extra state variable $y(t)$ representing the time:

$$y(t) = t, \qquad \text{with} \quad \dot{y}(t) = 1 \quad \text{and} \quad y(0) = 0.$$

- Convert the problem into standard form by introducing the extended state $\xi = [\,x,\, y\,]^T$:
  The dynamics read now as

$$\dot{\xi}(t) = \tilde{f}(\xi, u) = [\,f(x, u, y),\, 1\,]^T$$

  and the cost is defined by

$$\tilde{h}(\xi(T)) + \int_0^T \tilde{g}(\xi, u)\, dt,$$

  where $\tilde{g}(\xi, u) = g(x, u, y)$ and $\tilde{h}(\xi) = h(x)$.
  The Hamiltonian follows from above's definitions:

$$\tilde{H}(\xi, u, \tilde{p}) = \tilde{g}(\xi, u) + \tilde{p}^T \tilde{f}(\xi, u) \qquad \text{with} \;\; \tilde{p} = [\,p,\, p_y\,]^T.$$

- Apply the Minimum Principle:
  Denoting the optimal control by $u^*(t)$ and the corresponding optimal state by $\xi^*(t)$, we get the following:

  1. The adjoint equation is given by

$$\dot{\tilde{p}}(t) = -\frac{\partial \tilde{H}}{\partial \xi}\left(\xi^*(t), u^*(t), \tilde{p}(t)\right), \qquad \tilde{p}(T) = \frac{\partial \tilde{h}}{\partial \xi}\left(\xi^*(T)\right). \tag{2}$$

     However,

$$\tilde{H}(\xi, u, \tilde{p}) \;=\; g(x, u, y) + p^T f(x, u, y) + p_y \;=\; H(x, u, p, y) + p_y;$$

     that is,

$$\frac{\partial \tilde{H}}{\partial x} = \frac{\partial H}{\partial x}, \qquad \frac{\partial \tilde{H}}{\partial y} = \frac{\partial H}{\partial y}.$$

     Moreover,

$$\frac{\partial \tilde{h}}{\partial x} = \frac{\partial h}{\partial x} \quad \text{and} \quad \frac{\partial \tilde{h}}{\partial y} = 0.$$

     From (2), we recover the first equation

$$\dot{p}(t) = -\frac{\partial H}{\partial x}\left(x^*(t), u^*(t), p(t), t\right), \qquad p(T) = \frac{\partial h}{\partial x}\left(x^*(T)\right).$$

     In addition, replacing $y(t)$ by t again, we get

$$\dot{p}_y(t) = -\frac{\partial H}{\partial t}\left(x^*(t), u^*(t), p(t), t\right), \qquad p_y(T) = 0.$$

2. The optimal input $u^*(t)$ is obtained by

$$\begin{aligned} u^*(t) &= \arg\min_{u \in U} \left\{ H(x^*(t), u^*(t), p(t), t) + p_y(t) \right\} \\ &= \arg\min_{u \in U}\ H(x^*(t), u^*(t), p(t), t). \end{aligned}$$

3. Finally, the standard formulation gives us

$$H(x^*(t), u^*(t), p(t), t) + p_y(t) \text{ is constant.}$$

However, $p_y(t)$ is constant only if $\frac{\partial H}{\partial t} = 0$, which, in general, is only true if f and g do not depend on time.