Introduction to Recursive Filtering and Estimation

Spring 2010

Problem Set:
Particle Filter

Notes:

- **Notation:** The probability density function of a random variable w is denoted by $f_w(w)$ or the short hand $f(w)$. In the context of particle filter, the notation $x(n,k|k-1)$ is used to represent the *a priori* value of particle n at time k propagated forward in time using the process equations. Only measurements up to and including time $(k-1)$ are taken into account. After a measurement update with $z(k)$ and an appropriate resampling, the *a posteriori* particle at time k is obtained, denoted by $x(n,k|k)$.
- Please report any error that you may find to the teaching assistants (`strimpe@ethz.ch` or `aschoellig@ethz.ch`).

Problem Set

Problem 1

Suppose you have a measurement $z(k) = x(k)^2 + w(k)$, where $w(k)$ has a triangular probability density function that is given as

$$f(w(k)) = \begin{cases} 1/2 + w(k)/4 & \text{if } \ w(k) \in [-2, 0] \\ 1/2 - w(k)/4 & \text{if } \ w(k) \in [0, 2] \\ 0 & \text{otherwise}\,. \end{cases}$$

Suppose that five *a priori* particles $x(n, k|k-1),\ n = 1, 2, 3, 4, 5$, are given as -2, -1, 0, 1, and 2, and that the measurement is obtained as $z(k) = 1$. What are the weights β_n of the particles $x(n, k|k-1)$?

Problem 2

Suppose that five *a priori* particles $x(n, k|k-1),\ n = 1, 2, 3, 4, 5$, are found to have probabilities β_n of $0.1,\ 0.1,\ 0.1,\ 0.2$, and 0.5 given a measurement at time k. The particles are resampled with the basic strategy covered in class, where for a total number of N samples, the following two steps are repeated N times:

– Generate a random number r that is uniformly distributed on $[0, 1]$.

– Pick particle m such that $\sum_{n=1}^{m} \beta_n \geq r\,, \quad \sum_{n=1}^{m-1} \beta_n < r\,.$

a) What is the probability that the first particle will be chosen as an *a posteriori* particle at least once, i.e. $x(n, k|k) = x(1, k|k-1)$ for some $n \in \{1, 2, \ldots, 5\}$?

b) What is the probability that the fifth particle will be chosen as an *a posteriori* particle at least once, i.e. $x(n, k|k) = x(5, k|k-1)$ for some $n \in \{1, 2, \ldots, 5\}$?

c) What is the probability that the five *a posteriori* particles will be equal to the five *a priori* particles (disregarding order)?

Problem 3

In class, we introduced a roughening procedure that adds to each element $x_i(n, k|k),\ i \in \{1, 2, \ldots, d\}$, of the particle $x(n, k|k) \in \mathbb{R}^d$ a random variable with a standard deviation of $KE_iN^{-1/d}$, where K is a tuning parameter and N is the number of particles. The values E_i represent the maximum difference between the particle elements before roughening, i.e.

$$E_i = \max_{n_1, n_2} |x_i(n_1, k|k) - x_i(n_2, k|k)|\ .$$

Suppose that you have five particles -1, -1, 0, 1, and 1. You want to use the described roughening procedure and add a uniformly distributed random variable with the given standard deviation. What range of K will give a probability of at least $1/8$ that at least one of the roughened particles is less than -2?

Problem Set # 5

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Problem 1

$\beta_n = \alpha\, f(z(k) \,|\, x(n, k|k-1))$

Calculate for each particle: $w(k) = z(k) - x(k)^2$ and obtain $f(z(k) \,|\, x(n,k|k-1)$ from $f(w(k))$

n	$w(k)$	$f(w(k)) \hat{=} f(z(k) \,|\, x(n, k|k-1))$
1	$1-(-2)^2 = -3$	$f(-3) = 0$
2	$1-(-1)^2 = 0$	$f(0) = \frac{1}{2}$
3	$1- 0^2 = 1$	$f(1) = \frac{1}{4}$
4	$1 - 1^2 = 0$	$f(0) = \frac{1}{2}$
5	$1 - 2^2 = -3$	$f(-3) = 0$

$$\alpha = \left(\sum_{i=1}^{5} f(z(k) \,|\, x(n, k|k-1)) \right)^{-1}$$

$$= \frac{4}{5}$$

$\Rightarrow$ $\beta_1 = 0$ (for $x(1, k|k-1) = -2$)

$\beta_2 = \frac{2}{5}$ (" $x(2,$ " $) = -1$)

$\beta_3 = \frac{1}{5}$ (" $x(3,$ " $) = 0$)

$\beta_4 = \frac{2}{5}$ (" $x(4,$ " $) = 1$)

$\beta_5 = 0$ (" $x(5,$ " $) = 2$)

Problem 2

a) Pr(first particle is chosen at least once)

$= 1 - Pr(\text{first particle is never chosen})$

$= 1 - 0.9^5$

$\approx 0.4095 = 40.95\%$

b) Pr(fifth particle is chosen at least once)

$= 1 - 0.5^5$

$\approx 0.9688 = 96.88\%$

c) Pr(posterior particles equal to a prior ones)

$= 0.1 \cdot 0.1 \cdot 0.1 \cdot 0.2 \cdot 0.5 \cdot \underbrace{5!}_{\text{all possible combinations of rearranging}}$

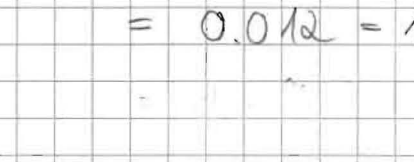
$= 0.012 = 1.2\%$

Problem 3

Here, $d = 1$, $N = 5$, $E_1 = 2$.

$\Rightarrow$ standard deviation $\sigma = \frac{2K}{5}$

For a general uniform distribution on the interval $[-\bar{s}, \bar{s}]$,



$\sigma = \frac{1}{\sqrt{3}} \bar{s}$.

$\Rightarrow \bar{s} = \frac{2\sqrt{3} K}{5}$

Call the roughened particles

$$x_n = x(n, k|k) + \underset{\substack{\uparrow \\ \text{uniformly distributed}}}{\Delta x(n,k)}$$

$$Pr(x_1 < -2) = Pr(\Delta x(n,k) < -1) = \max\left(0, \int_{-\bar{s}}^{-1} \frac{1}{2\bar{s}}\, ds\right)$$

$$= \max\left(0, \frac{-1+\bar{s}}{2\bar{s}}\right)$$

$$Pr(x_2 < -2) = \max\left(0, \frac{\bar{s}-1}{2\bar{s}}\right)$$
$$Pr(x_3 < -2) = \max\left(0, \frac{\bar{s}-2}{2\bar{s}}\right)$$
$$Pr(x_4 < -2) = \max\left(0, \frac{\bar{s}-3}{2\bar{s}}\right)$$
$$Pr(x_5 < -2) = \max\left(0, \frac{\bar{s}-3}{2\bar{s}}\right)$$

Consider four different cases:

(I) $\bar{s} \leq 1$

(II) $1 < \bar{s} \leq 2$

(III) $2 < \bar{s} \leq 3$

(IV) $\bar{s} > 3$

$$\Rightarrow Pr(\text{at least one} < -2)$$
$$= 1 - \prod_{i=1}^{5} Pr(x_i \geq -2)$$
$$= 1 - \prod_{i=1}^{5} \left(1 - Pr(x_i < -2)\right)$$

For (I):

$$Pr(\text{at least one} < -2) = 0$$

For (II):

Pr (at least one < -2)

$= 1 - \left(1 - \frac{\bar{s}-1}{2\bar{s}}\right)^2 \overset{!}{\geq} \frac{1}{8}$, $\bar{s} \in [1,2]$

Solving the quadratic equation and noticing that $\bar{s} \in [1,2]$

$\Rightarrow \bar{s} \geq \frac{2+\sqrt{14}}{5} \approx 1.15$

For (III):

Pr(at least one < -2)

$= 1 - \left(1 - \frac{\bar{s}-1}{2\bar{s}}\right)^2 \left(1 - \frac{\bar{s}-2}{2\bar{s}}\right)$, $\bar{s} \in [2,3]$

$\Rightarrow$ This is always larger than $\frac{1}{8}$.

For (IV):

Pr(at least one < -2)

$= 1 - \left(1 - \frac{\bar{s}-1}{2\bar{s}}\right)^2 \left(1 - \frac{\bar{s}-2}{2\bar{s}}\right)\left(1 - \frac{\bar{s}-3}{2\bar{s}}\right)$, $\bar{s} > 3$

$\Rightarrow$ This is always larger than $\frac{1}{8}$

$\Rightarrow$ To sum up:

For $\bar{s} \geq \frac{2+\sqrt{14}}{5} \iff \frac{2\sqrt{3}\,k}{\cancel{5}} \geq \frac{2+\sqrt{14}}{\cancel{5}}$

$\iff k \geq \frac{2+\sqrt{14}}{2\sqrt{3}} \approx 1.66$

the probability that at least one roughened particle is less than -2 is at least $\frac{1}{8}$.