Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute for Dynamic Systems and Control
Institut für Dynamische Systeme und Regelungstechnik

Introduction to Recursive Filtering and Estimation

Spring 2010

Problem Set: Kalman Filter

Notes:

- **Notation:** A scalar valued normally distributed random variable x with mean μ and variance σ^2 is denoted by $x \sim \mathcal{N}(\mu, \sigma^2)$; a vector valued normally distributed random variable x with mean μ and covariance matrix Σ is denoted by $x \sim \mathcal{N}(\mu, \Sigma)$.
- Please report any error that you may find to the teaching assistants (`strimpe@ethz.ch` or `aschoellig@ethz.ch`).

Problem Set

Problem 1

In class it was shown that for two scalar independent random variables $x \sim \mathcal{N}(\mu_x, \sigma_x^2)$, $y \sim \mathcal{N}(\mu_y, \sigma_y^2)$, and $z := x + y$,

$$f_x(x) f_y(z-x) \propto \exp\left[-\frac{1}{2} \underbrace{\left(\frac{1}{\sigma_x^2}(x-\mu_x)^2 + \frac{1}{\sigma_y^2}(z-x-\mu_y)^2 \right)}_{=:\xi(x,z)} \right],$$

where $\propto$ means proportional. Show that $\xi(x,z)$ can be written as follows:

$$\xi(x,z) = a\big(x-(b+cz)\big)^2 + d(z-e)^2 + f,$$

with real coefficients a, b, c, d, e, f and, in particular, $a > 0$, $d > 0$.

Problem 2

Using the notation and the result of Problem 1, it was shown in class that

$$f_z(z) \propto \int_{-\infty}^{\infty} \exp\Big(-\frac{a}{2}\big(x-(b+cz)\big)^2 \Big) \exp\Big(-\frac{d}{2}(z-e)^2 \Big)\, dx. \tag{1}$$

Prove that (1) implies

$$f_z(z) \propto \exp\Big(-\frac{d}{2}(z-e)^2 \Big).$$

Problem 3

For $x \sim \mathcal{N}(\mu_x, \sigma_x^2)$, $y \sim \mathcal{N}(\mu_y, \sigma_y^2)$, and $z = x + y$, it was shown in class (using the results of Problem 1 and 2) that z is a Gaussian random variable, i.e. $z \sim \mathcal{N}(\mu_z, \sigma_z^2)$. Compute μ_z and σ_z from μ_x, μ_y, σ_x, and σ_y.

Problem 4

For the linear discrete-time system

$$\begin{aligned} x(k) &= A(k)x(k-1) + B(k)u(k) + v(k) \\ z(k) &= H(k)x(k) + w(k), \end{aligned}$$

with $v(k) \sim \mathcal{N}(0, Q(k))$ and $w(k) \sim \mathcal{N}(0, R(k))$ and $\{x(0), v(1), \ldots, v(k), w(1), \ldots, w(k)\}$ independent, derive the equations of the prediction step (S1) of the Kalman filter,

$$\begin{aligned} \hat{x}(k|k-1) &= A(k)\hat{x}(k-1|k-1) + B(k)u(k) \\ P(k|k-1) &= A(k)P(k-1|k-1)A^T(k) + Q(k), \end{aligned}$$

where the notation used was introduced in class.

Problem 5

a) **(optional)** Prove the *matrix inversion lemma*: If A, D, and $D^{-1}+CA^{-1}B$ are nonsingular, then $A+BDC$ is nonsingular and

$$(A+BDC)^{-1} = A^{-1} - A^{-1}B(D^{-1}+CA^{-1}B)^{-1}CA^{-1}.$$

b) Using the matrix inversion lemma, prove the alternate form of the Kalman filter measurement update equations (S2) that have been introduced in class:

$$\begin{aligned}
K(k) &= P(k|k-1)\,H^T(k)\,\big(H(k)P(k|k-1)H^T(k)+R(k)\big)^{-1}\\
\hat{x}(k|k) &= \hat{x}(k|k-1) + K(k)\,\big(z(k) - H(k)\hat{x}(k|k-1)\big)\\
P(k|k) &= \big(I-K(k)H(k)\big)\,P(k|k-1)\,\big(I-K(k)H(k)\big)^T + K(k)\,R(k)\,K^T(k).
\end{aligned}$$

Problem 6

Show that the covariance matrix $P_\infty = \lim_{k\to\infty} P(k|k-1)$ of the steady state Kalman filter satisfies the discrete algebraic Riccati equation (DARE)

$$P_\infty = AP_\infty A^T - AP_\infty H^T\big(HP_\infty H^T + R\big)^{-1}HP_\infty A^T + Q.$$

Problem 7

A radioactive particle mass has a half-life of τ seconds. At each time step the number of emitted particles x is half of what it was one time step ago, but there is some error $v(k)$ (zero-mean with variance Q) in the number of emitted particles due to background radiation. At each time step, the number of emitted particles is counted. The instrument used to count the number of emitted particles has a random error at time k of $w(k)$, which is zero-mean with a variance R. Assume that $v(k)$ and $w(k)$ are uncorrelated.

a) Write the linear system equations for this system.

b) Design a Kalman filter to estimate the number of emitted particles at each time step.

c) What is the steady-state Kalman gain K when $Q=R$? What is the steady-state Kalman gain when $Q=2R$? Give an intuitive explanation for why the steady-state gain changes the way it does when the ratio of Q to R changes.

Problem 8

Suppose that you have a fish tank with x_p piranhas and x_g guppies. Once per week, you put guppy food into the tank (which the piranhas do not eat). Each week the piranhas eat some of the guppies. The birth rate of the piranhas is proportional to the guppy population, and the death rate of the piranhas is proportional to their own population (due to overcrowding). Therefore $x_p(k+1) = x_p(k) + k_1 x_g(k) - k_2 x_p(k) + v_p(k)$, where k_1 and k_2 are proportionality constants and $v_p(k)$ is white noise with a variance of one that accounts for mismodeling. The birth rate of the guppies is proportional to the food supply u, and the death rate of the guppies is proportional to the piranha population. Therefore, $x_g(k+1) = x_g(k) + u(k) - k_3 x_p(k) + v_g(k)$, where k_3 is a proportionality constant and $v_g(k)$ is white noise with a variance of one that accounts for mismodeling. The step size for this model is one week. Every week, you count the piranhas and guppies. You can count the piranhas accurately because they are so large, but your guppy count has zero-mean noise with a variance of one. Assume that $k_1 = 1$ and $k_2 = k_3 = 1/2$.

a) Generate a linear state-space model for this system.

b) Suppose that at the initial time you have a perfect count for x_p and x_g. Using a Kalman filter to estimate the guppy population, what is the variance of your guppy population estimate after one week? What is the variance after two weeks?

c) What is the ratio of the expected piranha population to the expected guppy population when they reach steady state?

Problem Set 4: Recursive Filtering and Estimation

Kalman Filter

Problem 1

- $\left(\frac{1}{\sigma_x^2}(x-\mu_x)^2 + \frac{1}{\sigma_y^2}(z-x-\mu_y)^2\right)$

$$= \frac{1}{\sigma_x^2}x^2 - 2\frac{\mu_x}{\sigma_x^2}x + \frac{\mu_x^2}{\sigma_x^2} + \frac{1}{\sigma_y^2}z^2 + \frac{1}{\sigma_y^2}x^2 + \frac{\mu_y^2}{\sigma_y^2} - \frac{2}{\sigma_y^2}xz - \frac{2\mu_y}{\sigma_y^2}z + \frac{2\mu_y}{\sigma_y^2}x$$

$$= \left(\frac{1}{\sigma_x^2} + \frac{1}{\sigma_y^2}\right)x^2 + \frac{1}{\sigma_y^2}z^2 - \frac{2}{\sigma_y^2}xz - 2\left(\frac{\mu_x}{\sigma_x^2} - \frac{\mu_y}{\sigma_y^2}\right)x - 2\frac{\mu_y}{\sigma_y^2}z + \frac{\mu_x^2}{\sigma_x^2} + \frac{\mu_y^2}{\sigma_y^2}$$

- $a(x-(b+cz))^2 + d(z-e)^2 + f$

$$= a(x^2 - 2bx - 2cxz + b^2 + 2bcz + c^2z^2) + d(z^2 - 2ez + e^2) + f$$

$$= ax^2 + (ac^2+d)z^2 - 2acxz - 2abx + (2abc - 2de)z + ab^2 + de^2 + f$$

- Equating coefficients:

$a = \frac{1}{\sigma_x^2} + \frac{1}{\sigma_y^2} > 0.$ (coefficients of x^2)

$b = \frac{1}{a}\left(\frac{\mu_x}{\sigma_x^2} - \frac{\mu_y}{\sigma_y^2}\right)$ (coeff. x)

$c = \frac{1}{a}\cdot\frac{1}{\sigma_y^2}$ (coeff. xz)

$d = \frac{1}{\sigma_y^2} - ac^2$ (coeff. z^2)

$$= \frac{1}{\sigma_y^2} - \frac{1}{a}\frac{1}{\sigma_y^4} = \frac{1}{\sigma_y^2}\left(1 - \frac{1}{\sigma_y^2}\left(\frac{1}{\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2}}\right)\right) = \frac{1}{\sigma_y^2}\underbrace{\left(1 - \underbrace{\left(\frac{1}{1+\frac{\sigma_y^2}{\sigma_x^2}}\right)}_{<1}\right)}_{>0}$$

$\Rightarrow d > 0$

$e = \left(abc + \frac{\mu_y}{\sigma_y^2}\right)\frac{1}{d}$

$f = \frac{\mu_x^2}{\sigma_x^2} + \frac{\mu_y^2}{\sigma_y^2} - ab^2 - de^2$

Problem 2

- $f_z(z) \propto \int_{-\infty}^{\infty} \exp\left(-\frac{a}{2}(x-(b+cz))^2\right) \exp\left(-\frac{d}{2}(z-e)^2\right) dx$
  (see lecture notes)

- $\int_{-\infty}^{\infty} \exp\left(-\frac{a}{2}(x-(b+cz))^2\right) \exp\left(-\frac{d}{2}(z-e)^2\right) dx$

  $= \underbrace{\int_{-\infty}^{\infty} \exp\left(-\frac{a}{2}(x-(b+cz))^2\right) dx}_{C} \cdot \exp\left(-\frac{d}{2}(z-e)^2\right)$

- Show that C does not depend on z

  Substitution: $\bar{x} := x-(b+cz) \Rightarrow d\bar{x} = dx$

  $x \to \infty \Rightarrow \bar{x} \to \infty$, $x \to -\infty \Rightarrow \bar{x} \to -\infty$

  $\Rightarrow C = \int_{-\infty}^{\infty} \exp\left(-\frac{a}{2}(x-(b+cz))^2\right) dx = \int_{-\infty}^{\infty} \exp\left(-\frac{a}{2}\bar{x}^2\right) d\bar{x}$

  independent of z.

  $\therefore f_z(z) \propto \exp\left(-\frac{d}{2}(z-e)^2\right)$

Problem 3

- $\underline{\mu_z} = E[z] = E[x+y] = E[x] + E[y] = \underline{\mu_x + \mu_y}$

- $\underline{\underline{\sigma_z^2}} = E\left[(z-\mu_z)^2\right] = E\left[(x-\mu_x+y-\mu_y)^2\right]$

  $= E\left[(x-\mu_x)^2\right] + 2E\left[(x-\mu_x)(y-\mu_y)\right] + E\left[(y-\mu_y)^2\right]$

  $= \sigma_x^2 + \sigma_y^2 + 2\cdot(E[x]-\mu_x)(E[y]-\mu_y)$

  (by independence of x, y

  $= \underline{\underline{\sigma_x^2 + \sigma_y^2}}$

Problem 4

- $$\hat{x}(k|k-1) = E[x(k|k-1)]$$
$$= E[A(k)\, x(k-1|k-1) + B(k)\, u(k) + v(k)]$$
$$= A(k) \cdot E[x(k-1|k-1)] + B(k)\, E[u(k)] + E[v(k)]$$
$$= A(k)\, \hat{x}(k-1|k-1) + B(k) \cdot u(k)$$

- $$P(k|k-1) = E\left[(x(k|k-1) - \hat{x}(k|k-1))\,(x(k|k-1) - \hat{x}(k|k-1))^T \right]$$
$$= E\Big[\big(A(k)(x(k-1|k-1) - \hat{x}(k-1|k-1)) + v(k)\big) \cdot \big(A(k)(x(k-1|k-1) - \hat{x}(k-1|k-1)) + v(k)\big)^T \Big]$$
$$= A(k) \cdot E\left[(x(k-1|k-1) - \hat{x}(k-1|k-1))(x(k-1|k-1) - \hat{x}(k-1|k-1))^T \right] A^T(k) + E[v(k)\, v^T(k)]$$
(cross-terms are zero, since $x(k-1|k-1)$ and $v(k)$ independent)
$$= A(k) \cdot P(k-1|k-1) \cdot A^T(k) + Q(k)$$

Problem 5

a). The matrix inversion lemma can be proven by solving the following matrix equation:

$$\begin{bmatrix} A & B \\ C & -D^{-1} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

- Expanding yields

$$AX + BY = I \qquad (1)$$
$$CX - D^{-1}Y = 0 \qquad (2)$$

- We can rewrite (2): $D^{-1}Y = CX \Leftrightarrow Y = DCX$ (since D^{-1} invertible).
Plugging into (1) we obtain $(A + BDC)X = I$ and
$$X = (A + BDC)^{-1} \qquad (3)$$

- The other way around, we may solve (1) for X,

$$X = A^{-1}(I - BY) \qquad \text{(since } A \text{ invertible)}$$

and substitute in (2)

$$CA^{-1}(I - BY) = D^{-1}Y$$
$$\Leftrightarrow \quad CA^{-1} - CA^{-1}BY = D^{-1}Y$$
$$\Leftrightarrow \quad CA^{-1} = (D^{-1} + CA^{-1}B)Y$$
$$\Leftrightarrow \quad (D^{-1} + CA^{-1}B)^{-1} CA^{-1} = Y$$

Substituting this into (1), we have another equation for X:

$$AX + B(D^{-1} + CA^{-1}B)^{-1}CA^{-1} = I$$
$$\Leftrightarrow \quad X = A^{-1} - A^{-1}B(D^{-1} + CA^{-1}B)^{-1}CA^{-1} \qquad (4)$$

- equating (3) and (4), the result follows:

$$(A + BDC)^{-1} = A^{-1} - A^{-1}B(D^{-1} + CA^{-1}B)^{-1}CA^{-1}$$

- Note that if $A, D, D^{-1} + CA^{-1}B$ are nonsingular, then $A + BDC$ is nonsingular, since one can write

$$\begin{bmatrix} A & B \\ C & -D^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & -D^{-1} - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} I & -BD \\ 0 & I \end{bmatrix} \begin{bmatrix} A + BDC & 0 \\ 0 & -D^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -DC & I \end{bmatrix}$$

and therefore

$$\det(A) \cdot \det(-D^{-1} - CA^{-1}B) = \det(A + BDC) \cdot \det(-D^{-1}).$$

5)

- To simplify notation drop all "k" arguments and define $\bar{P} := P(k|k)$ and $P := P(k|k-1)$.

- To show that the update equations for $\hat{x}(k|k)$ are the same, it suffices to show that $K = \bar{P}H^T R^{-1}$.

$$\bar{P}HR = (P^{-1} + H^T R^{-1} H)^{-1} H^T R^{-1} \qquad \text{(by definition)}$$
$$= (P - PH^T(R + HPH^T)^{-1}HP)H^T R^{-1} \qquad \text{(by matrix inversion lemma)}$$
$$= PH^T(I - (R + HPH^T)^{-1}HPH^T)R^{-1}$$
$$= PH^T(R + HPH^T)^{-1}(R + HPH^T - HPH^T)R^{-1}$$

$$= PH^T(R + HPH^T)^{-1}$$
$$= K \quad \text{(by definition)}$$

as required

- Update equation for $\bar{P}$:

$$\bar{P} = (P^{-1} + H^T R^{-1} H)^{-1}$$
$$= P - PH^T(R + HPH^T)^{-1} HP \quad \text{(by matrix inv. lemma)}$$
$$= P - KHP$$
$$= (I - KH)P$$
$$= (I - KH)P - (I - KH)PH^T K^T + (I - KH)PH^T K^T$$
$$= (I - KH)P(I - KH)^T + (PH^T - KHPH^T)K^T$$
$$= (I - KH)P(I - KH)^T + (PH^T - K(HPH^T + R) + KR)K^T$$
$$= (I - KH)P(I - KH)^T + (PH^T - PH^T + KR)K^T$$
(by def. of K)
$$= (I - KH)P(I - KH)^T + KRK^T$$

q.e.d.

Problem 6

- KF equations:

$$P(k|k-1) = A(k)\,P(k-1|k-1)\,A^T(k) + Q(k) \quad (1)$$
$$P(k|k) = \left(P^{-1}(k|k-1) + H^T(k)\,R^{-1}(k)\,H(k)\right)^{-1} \quad (2)$$

- Steady-state KF: $A(k) = A$, $H(k) = H$, $R(k) = R$, $Q(k) = Q$

$P_\infty := \lim_{k \to \infty} P(k|k-1)$.

→ insert (2) into (1), rewrite:

$$P_\infty = A\left(P_\infty^{-1} + H^T R^{-1} H\right)^{-1} A^T + Q$$

- Apply Matrix inversion lemma from Problem 5a)

$$P_\infty = A\left(P_\infty - P_\infty H^T\left(R + HP_\infty H^T\right)^{-1} HP_\infty\right)A^T + Q$$

$$P_\infty = AP_\infty A^T - AP_\infty H^T(R + HP_\infty H^T)^{-1} HP_\infty A^T + Q.$$

q.e.d.

Problem 7

a) System equations:

$$x(k+1) = \frac{1}{2} x(k) + v(k) \qquad E[v(k)] = 0, \ Var(v(k)) = Q$$
$$z(k) = x(k) + w(k) \qquad E[w(k)] = 0, \ Var(w(k)) = R$$

b) Kalman Filter

Step 1:

$$\hat{x}(k|k-1) = \frac{1}{2} \hat{x}(k-1|k-1)$$
$$P(k|k-1) = \frac{1}{4} P(k-1|k-1) + Q$$

Step 2:

$$K(k) = P(k|k-1)\left(P(k|k-1) + R\right)^{-1} = \frac{P(k|k-1)}{P(k|k-1) + R}$$

$$\hat{x}(k|k) = \hat{x}(k|k-1) + K(k)\left(z(k) - \hat{x}(k|k-1)\right)$$

$$P(k|k) = (1 - K(k))\, P(k|k-1)\, (1 - K(k)) + K(k)\, R\, K(k)$$
$$= (1 - K(k))^2\, P(k|k-1) + K(k)^2 \cdot R$$

Initialization:

$\hat{x}(0|0) = x_0$, $P(0|0) = P_0$ not given in problem

c) Steady-state Kalman Filter

- obtain steady-state estimation variance P_∞ from discrete algebraic Riccati equation:

$$P_\infty = A P_\infty A^T - A P_\infty H^T (H P_\infty H^T + R)^{-1} H P_\infty A^T + Q$$
$$= \frac{1}{4} P_\infty - \frac{1}{4} P_\infty^2 (P_\infty + R)^{-1} + Q$$
$$\Leftrightarrow \frac{3}{4} P_\infty^2 + \frac{3}{4} P_\infty R = Q P_\infty + R \cdot Q - \frac{1}{4} P_\infty^2$$
$$\Leftrightarrow P_\infty^2 + \left(\frac{3}{4} R - Q\right) P_\infty - RQ = 0$$

$$P_\infty = -\frac{1}{2}\left(\frac{3}{4} R - Q\right) \pm \frac{1}{2} \sqrt{\left(\frac{3}{4} R - Q\right)^2 + 4RQ}$$

- $Q = R$: $P_\infty = \left(\frac{1}{8} \pm \frac{1}{2}\sqrt{\left(\frac{1}{4}\right)^2 + 4}\right) R$

$= 1.133 R$ or -0.883

(only positive solution makes sense!)

$\rightarrow \underline{\underline{K}} = \frac{P_\infty}{P_\infty + R} = \frac{1.133}{2.133} = \underline{\underline{0.53}}$

- $Q = 2R$: $P_\infty = \left(\frac{5}{8} \pm \frac{1}{2}\sqrt{\left(\frac{5}{4}\right)^2 + 8}\right) R$

$= 2.171 R$

$\rightarrow \underline{\underline{K = 0.68}}$

- In general, as the process noise increases relative to the measurement noise, the gain K increases, i.e. the filter puts more emphasis on the measurements rather than the prediction due to the model.

Problem 8

a) - $x(k) = \begin{bmatrix} x_p(k+1) \\ x_g(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 1-k_2 & k_1 \\ -k_3 & 1 \end{bmatrix}}_{=:A} \begin{bmatrix} x_p(k-1) \\ x_g(k-1) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{=:B} u(k) + v(k)$

with $E[v(k)] = 0$, $Var(v(k)) = E[v(k)\, v^T(k)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = Q$

$z(k) = x(k) + w(k)$

with $E[w(k)] = 0$, $Var(w(k)) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = R$

- With numeric values:

$$x(k) = \begin{bmatrix} 0.5 & 1 \\ -0.5 & 1 \end{bmatrix} x(k-1) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) + v(k)$$

$$z(k) = x(k) + w(k)$$

b) · initialization of KF: $P(0|0) = 0$

· variance after one week: $P(1|1) = ?$

Using the KF equations:

S1: $P(1|0) = A P(0|0) A^T + Q = Q = I$

S2: $K(1) = I \cdot (I + R)^{-1}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$P(1|1) = (I - K(1) I) \cdot P(1|0) (I - K(1) I)^T + K(1) \cdot R \; K^T(1)$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

-> the variance of the guppy population is $\frac{1}{2}$ after 1 week.

· variance after two weeks: $P(2|2) = ?$

S1: $$P(2|1) = \begin{bmatrix} 0.5 & 1 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0.5 & -0.5 \\ 1 & 1 \end{bmatrix} + I$$

$$= \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0.5 & -0.5 \\ 1 & 1 \end{bmatrix} + I = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$$

S2: $$K(2) = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \left(\frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 0 \\ 1/7 & 4/7 \end{bmatrix}$$

$$P(2|2) = \begin{bmatrix} 0 & 0 \\ -\frac{1}{7} & \frac{3}{7} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{7} \\ 0 & \frac{3}{7} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \frac{1}{7} & \frac{4}{7} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{7} \\ 0 & \frac{4}{7} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & \frac{4}{7} \end{bmatrix}$$

-> the variance of the guppy population after 2 weeks is $\frac{4}{7}$.

c) · Steady-state: $E[x(k+1)] = E[x(k)] =: \bar{x}$

$\Rightarrow \bar{x} = E[x(k+1)] = A \cdot E[x(k)] + B u(k) = A\bar{x} + Bu$

$$\bar{x} = (I - A)^{-1} B u = \begin{bmatrix} 0.5 & -1 \\ 0.5 & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$

$\Rightarrow \bar{x}_p / \bar{x}_g = 2/1$.