ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute for Dynamic Systems and Control
IDSC
Institut für Dynamische Systeme und Regelungstechnik

Introduction to Recursive Filtering and Estimation

Spring 2010

Problem Set:

Bayes Theorem,
recursive estimation using Bayes Theorem

Notes:

- **Notation:** Unless otherwise noted, x, y, and z denote random variables, $f_x(x)$ (or the short hand $f(x)$) denotes the probability density function of x, and $f_{x|y}(x|y)$ (or $f(x|y)$) denotes the conditional probability density function of x conditioned on y. The expected value is denoted by $\mathrm{E}[\cdot]$, the variance is denoted by $\mathrm{Var}(\cdot)$ and $\Pr(Z)$ denotes the probability that the event Z occurs.
- Please report any error that you may find to the teaching assistants (`strimpe@ethz.ch` or `aschoellig@ethz.ch`).

Problem Set

Problem 1

Mr. Jones has devised a gambling system for winning at roulette. When he bets, he bets on red, and places a bet only when the ten previous spins of the roulette have landed on a black number. He reasons that his chance of winning is quite large since the probability of eleven consecutive spins resulting in black is quite small. What do you think of this system?

Problem 2

Consider two boxes, one containing one black and one white marble, the other, two black and one white marble. A box is selected at random and a marble is drawn at random from the selected box. What is the probability that the marble is black?

Problem 3

In Problem 2, what is the probability that the first box was the one selected given that the marble is white?

Problem 4

Urn 1 contains two white balls and one black ball, while urn 2 contains one white ball and five black balls. One ball is drawn at random from urn 1 and placed in urn 2. A ball is then drawn from urn 2. It happens to be white. What is the probability that the transferred ball was white?

Problem 5

Stores A, B and C have 50, 75, 100 employees, and respectively 50, 60 and 70 percent of these are women. Resignations are equally likely among all employees, regardless of sex. One employee resigns and this is a woman. What is the probability that she works in store C?

Problem 6

a) A gambler has in his pocket a fair coin and a two-headed coin. He selects one of the coins at random, and when he flips it, it shows heads. What is the probability that it is the fair coin?

b) Suppose that he flips the same coin a second time and again it shows heads. What is now the probability that it is the fair coin?

c) Suppose that he flips the same coin a third time and it shows tails. What is now the probability that it is the fair coin?

Problem 7

Urn 1 has five white and seven black balls. Urn 2 has three white and twelve black balls. We flip a fair coin. If the outcome is heads, then a ball from urn 1 is selected, while if the outcome is tails, then a ball from urn 2 is selected. Suppose that a white ball is selected. What is the probability that the coin landed tails?

Problem 8

An urn contains b black balls and r red balls. One of the balls is drawn at random, but when it is put back in the urn c additional balls of the same color are put in with it. Now suppose that we draw another ball. Show that the probability that the first ball drawn was black given that the second ball drawn was red is $b/(b+r+c)$.

Problem 9

Three prisoners are informed by their jailer that one of them has been chosen at random to be executed, and the other two are to be freed. Prisoner A asks the jailer to tell him privately which of his fellow prisoners will be set free, claiming that there would be no harm in divulging this information, since he already knows that at least one will go free. The jailer refuses to answer this question, pointing out that if A knew which of his fellows were to be set free, then his own probability of being executed would rise from 1/3 to 1/2, since he would then be one of two prisoners. What do you think of the jailer's reasoning?

Problem 10

Let x and y be independent random variables. Let $g(\cdot)$ and $h(\cdot)$ be arbitrary functions of x and y, respectively. Define the random variables $v = g(x)$ and $w = h(y)$. Prove that v and w are independent. That is, functions of independent random variables are independent.

Problem 11

Let x be a continuous, uniformly distributed random variable with $x \in \mathcal{X} = [-5, 5]$. Let

$$\begin{aligned} z_1 &= x + n_1 \\ z_2 &= x + n_2, \end{aligned}$$

where n_1 and n_2 are continuous random variables with probability density functions

$$f(n_1) = \begin{cases} \alpha_1\,(1+n_1) & \text{for } -1 \leq n_1 \leq 0 \\ \alpha_1\,(1-n_1) & \text{for } \quad 0 \leq n_1 \leq 1 \\ 0 & \text{otherwise}, \end{cases}$$

$$f(n_2) = \begin{cases} \alpha_2\left(1+\frac{1}{2}n_2\right) & \text{for } -2 \leq n_2 \leq 0 \\ \alpha_2\left(1-\frac{1}{2}n_2\right) & \text{for } \quad 0 \leq n_2 \leq 2 \\ 0 & \text{otherwise}, \end{cases}$$

where α_1 and α_2 are normalization constants. Assume that the random variables x, n_1, n_2 are independent, i.e. $f(x, n_1, n_2) = f(x)f(n_1)f(n_2)$.

a) Calculate α_1 and α_2.

b) Calculate $f(x|z_1 = 0, z_2 = 0)$.

c) Calculate $f(x|z_1 = 0, z_2 = 1)$.

d) Calculate $f(x|z_1 = 0, z_2 = 3)$.

Discuss the results.

Problem 12

Consider the following estimation problem: an object B moves randomly on a circle with radius 1. The distance to the object can be measured from a given observation point P. The goal is to estimate the location of the object, see Figure 1.

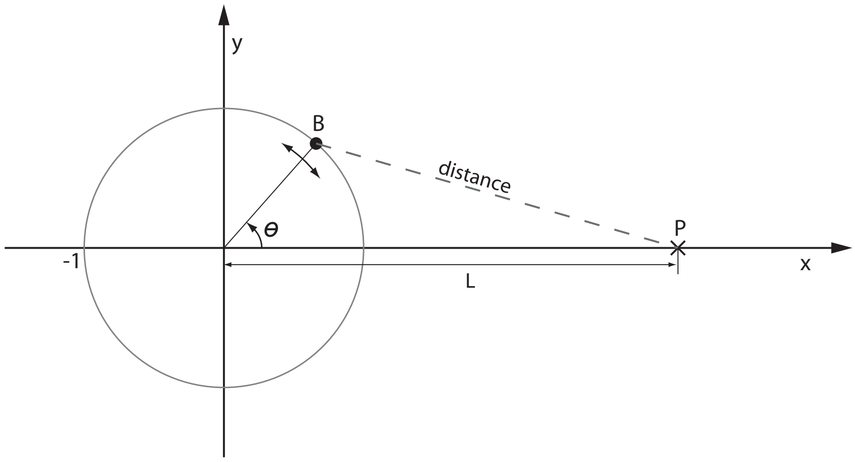


Figure 1

The object B can only move in discrete steps. The object's location at time k is given by $x(k) \in \{0, 1, \ldots, N-1\}$, where

$$\theta(k) = 2\pi \frac{x(k)}{N}.$$

The dynamics are

$$x(k) = \mod\left(x(k-1) + v(k)\,,\, N\right), \qquad k = 1, 2, \ldots,$$

where $v(k) = 1$ with probability p and $v(k) = -1$ otherwise. Note that $\mod(N,\, N) = 0$ and $\mod(-1,\, N) = N - 1$. The distance sensor measures

$$z(k) = \left((L - \cos\theta\,(k))^2 + (\sin\theta\,(k))^2 \right)^{\frac{1}{2}} + w(k),$$

where $w(k)$ represents the sensor error which is uniformly distributed on $[-e, e]$. We assume that $x(0)$ is uniformly distributed and $x(0)$, $v(k)$ and $w(k)$ are independent.

Simulate object movement and implement a Bayesian tracking algorithm that calculates for each time step k the probability density function $f(x(k)|z(1:k))$.

a) Test the following settings and discuss the results: $N = 100$, $x(0) = \frac{N}{4}$, $e = 0.5$,

$$\begin{aligned} L &= 2, & p &= 0.5, \\ L &= 2, & p &= 0.55, \\ L &= 0.1, & p &= 0.55, \\ L &= 0, & p &= 0.55. \end{aligned}$$

b) How robust is the algorithm? Set $N = 100$, $x(0) = \frac{N}{4}$, $e = 0.5$, $L = 2$, $p = 0.55$ in the simulation, but use slightly different values for p and e in your estimation algorithm, $\hat{p}$ and $\hat{e}$, respectively. Test the algorithm and explain the result for:

$$\begin{aligned} \hat{p} &= 0.45, & \hat{e} &= e, \\ \hat{p} &= 0.5, & \hat{e} &= e, \\ \hat{p} &= 0.9, & \hat{e} &= e, \\ \hat{p} &= p, & \hat{e} &= 0.9, \\ \hat{p} &= p, & \hat{e} &= 0.45. \end{aligned}$$

Problem Set #2

- Angela Schölig, aschoellig@ethz.ch -

Problem 1

- x_i : discrete random variable representing outcome of ith spin, $x_i \in \{red, black\}$, assume both equally likely
- Assuming <u>independence</u> between spins;
  $Pr(x_k, x_{k-1}, x_{k-2}, \dots, x_1) = P(x_k) \cdot P(x_{k-1}) \dots P(x_1)$,
  the probability of 11 consecutive spins resulting in black
  $Pr(x_{11} = black, x_{10} = black, \dots x_1 = black) = \left(\frac{1}{2}\right)^{11}$
  is actually quite small.
- However, given that the previous 10 were black,
  $$Pr(x_{11} = black \mid x_{10} = black, x_9 = black, \dots x_1 = black) \underset{\substack{\uparrow \\ \text{independence}}}{=} Pr(x_{11} = black)$$
  $$= \underline{\underline{\frac{1}{2}}}$$
  $$= P(x_{11} = red \mid x_{10} = black, x_9 = black, \dots x_1 = black)$$

  i.e. it is equally likely that 10 blacks in a row are followed by red or followed by black (independence assumption)

  Example: two spins, combinations (red, red), (red, black), (black, red), (black, black) all equally likely

  $$P(x_2 = black \mid x_1 = red) = \frac{1}{2} \begin{array}{l} \leftarrow \text{(black, red)} \\ \leftarrow \text{(black, red), (red, red)} \end{array}$$

$\Rightarrow$ bad strategy :)

Problem 2

- $x \in \{1, 2\}$: discrete random variable representing probability of choosing box 1 or 2, $f_x(1) = f_x(2) = \frac{1}{2}$
- $y \in \{b, w\}$: discrete random variable representing probability of drawing a black or white marble

$$f_{y|x}(b|1) = \frac{1}{2} = f_{y|x}(w|1)$$

$$f_{y|x}(b|2) = \frac{2}{3} \qquad f_{y|x}(w|2) = \frac{1}{3}$$

- $$f_y(b) = f_{y|x}(b|1) \cdot f_x(1) + f_{y|x}(b|2) \cdot f_x(2)$$
  $$= \frac{1}{2} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} = \underline{\underline{\frac{7}{12}}}$$

  total probability theorem
  or
  marginal probability

Problem 3

$$f_{x|y}(1|w) = \frac{f_{y|x}(w|1) \cdot f_x(1)}{f_y(w)} = \frac{\frac{1}{2} \cdot \frac{1}{2}}{1 - f_y(b)} = \frac{\frac{1}{4}}{\frac{5}{12}}$$

$$= \underline{\underline{\frac{3}{5}}}$$

Problem 4

- Urn 1 : $x \in \{b, w\}$ $\quad f_x(b) = \frac{1}{3}$, $f_x(w) = \frac{2}{3}$
- Urn 2 : $y \in \{b, w\}$

  consider transfer:

  $f_{y|x}(b|b) = \frac{6}{7}$ $\qquad$ $f_{y|x}(b|w) = \frac{5}{7}$

  $f_{y|x}(w|b) = \frac{1}{7}$ $\qquad$ $f_{y|x}(w|w) = \frac{2}{7}$

- question: $f_{x|y}(w|w)$?

$$f_{x|y}(w|w) = \frac{f_{y|x}(w|w)\, f_x(w)}{f_y(w)} = \frac{\frac{2}{7} \cdot \frac{2}{3}}{f_{y|x}(w|b)\, f_x(b) + f_{y|x}(w|w) \cdot f_x(w)}$$

total probability

$$= \frac{\frac{2}{7} \cdot \frac{2}{3}}{\frac{1}{7} \cdot \frac{1}{3} + \frac{2}{7} \cdot \frac{2}{3}} = \underline{\underline{\frac{4}{5}}}$$

Problem 5

- $X \in \{A, B, C\}$: discrete random variable representing the probability of working in store A, B, C

$$f_X(A) = \frac{50}{225} = \frac{2}{9}$$
$$f_X(B) = \frac{75}{225} = \frac{1}{3}$$
$$f_X(C) = \frac{100}{225} = \frac{4}{9}$$

- $y \in \{m, f\}$: discrete random variable representing the probability of being a woman

$$f_{Y|X}(f|A) = \frac{1}{2}$$
$$f_{Y|X}(f|B) = \frac{3}{5}$$
$$f_{Y|X}(f|C) = \frac{7}{10}$$

- question : $f_{X|Y}(C|f)$?

$$f_{X|Y}(C|f) = \frac{f_{Y|X}(f|C) \cdot f_X(C)}{f_Y(f)} = \frac{\frac{7}{10} \cdot \frac{4}{9}}{\sum_{i=A,B,C} f_{Y|X}(f|i) f_X(i)}$$

$$= \frac{\frac{7 \cdot 4}{90}}{\frac{1}{2} \cdot \frac{2}{9} + \frac{3}{5} \cdot \frac{1}{3} + \frac{7}{10} \cdot \frac{4}{9}} = \underline{\underline{\frac{1}{2}}}$$

Problem 6

(a)
- $x \in \{f, u\}$: fair or unfair coin, $f_x(f) = f_x(u) = \frac{1}{2}$
- $y \in \{h, t\}$: head or tail,

$$f_{y|x}(h|f) = \frac{1}{2} = f_{y|x}(t|f)$$
$$f_{y|x}(h|u) = 1 \qquad f_{y|x}(t|u) = 0$$

- question: $f_{x|y}(f|h)$?

$$f_{x|y}(f|h) = \frac{f_{y|x}(h|f) \cdot f_x(f)}{f_y(h)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\underbrace{f_{y|x}(h|f) \cdot f_x(f)}_{\text{fair coin}} + \underbrace{f_{y|x}(h|u) \cdot f_x(u)}_{\text{two-headed coin}}}$$

$$= \frac{\frac{1}{4}}{\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2}} = \underline{\underline{\frac{1}{3}}}$$

(b)
- y_1 denotes first flip, y_2 the second
- assume independence between both flips

$$f_{y_1,y_2|x}(y_1, y_2|x) = f_{y_1|x}(y_1|x) \cdot f_{y_2|x}(y_2|x)$$

- question: $f_{x|y_1,y_2}(f|h,h)$?

$$f_{x|y_1,y_2}(f|h,h) = \frac{f_{y_1,y_2|x}(h,h|f) \cdot f_x(f)}{f_{y_1,y_2}(h,h)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}}{\underbrace{\left(\frac{1}{2}\right)^2 \cdot \frac{1}{2}}_{\text{fair coin}} + \underbrace{\frac{1}{2}}_{\text{two-headed coin}}} = \underline{\underline{\frac{1}{5}}}$$

(c) Obviously, then, the probability is $\underline{1}$

Problem 7

Similar to Problem 2

- urn: $x \in \{1,2\}$ with $f_x(1) = f_x(2) = \frac{1}{2}$
- color: $y \in \{b, w\}$

$$f_{y|x}(b|1) = \frac{7}{12} \qquad f_{y|x}(w|1) = \frac{5}{12}$$
$$f_{y|x}(b|2) = \frac{12}{15} \qquad f_{y|x}(w|2) = \frac{3}{15}$$

- question: $f_{x|y}(2|w)$?

$$f_{x|y}(2|w) = \frac{f_{y|x}(w|2)\ f_x(2)}{f_y(w)}$$

$$= \frac{\frac{3}{15} \cdot \frac{1}{2}}{\frac{5}{12} \cdot \frac{1}{2} + \frac{3}{15} \cdot \frac{1}{2}} = \underline{\underline{\frac{12}{37}}}$$

Problem 8

- first ball: $x_1 \in \{b, r\}$, $f_{x_1}(b) = \frac{b}{b+r}$ $\quad f_{x_1}(r) = \frac{r}{b+r}$
- second ball: $x_2 \in \{b, r\}$

$$f_{x_2|x_1}(b|r) = \frac{b}{b+r+c} \qquad f_{x_2|x_1}(r|r) = \frac{c+r}{b+r+c}$$

$$f_{x_2|x_1}(b|b) = \frac{b+c}{b+r+c} \qquad f_{x_2|x_1}(r|b) = \frac{r}{b+r+c}$$

- question: $f_{x_1|x_2}(b|r)$?

$$f_{x_1|x_2}(b|r) = \frac{f_{x_2|x_1}(r|b)\ f_{x_1}(b)}{f_{x_2}(r)}$$

$$= \frac{\frac{r}{b+r+c} \cdot \frac{b}{b+r}}{\underbrace{\frac{c+r}{b+r+c} \cdot \frac{r}{b+r}}_{\text{first red}} + \underbrace{\frac{r}{b+r+c} \cdot \frac{b}{b+r}}_{\text{first black}}} \Rightarrow \underline{\underline{\frac{b}{b+r+c}}} \quad \text{q.e.d}$$

Problem 9

I) Popular, descriptive solution:

The probability that A is chosen to be executed is $\frac{1}{3}$ and there is a chance of $\frac{2}{3}$ that one of the others was chosen. If the jailer gives the name of one of the fellow prisoners who will be set free, prisoner A does not get new information about his own fate, but the probability of the remaining prisoner (B or C) to be executed is $\frac{2}{3}$ now. The probability of A being executed is still $\frac{1}{3}$.

II) Bayesian analysis:

- prisoner to be executed: $x \in \{A, B, C\}$
  → assumption of a random choice
  $$f_x(A) = f_x(B) = f_x(C) = \frac{1}{3}$$
- name of the prisoner which is given away by the jailer:
  $y \in \{A, B, C\}$
  → conditional probabilities

$$f_{y|x}(y|x) = \begin{cases} 0 & \text{if } x = y \quad \text{jailer does not lie} \\ 0 & \text{if } y = A \quad \text{jailer does not reveal asking person} \\ \frac{1}{2} & \text{if } x = A \quad \text{A the one executed, jailer mention B, C with equal probability} \\ 1 & \text{if } x \neq A \quad \text{jailer is forced to give away the name of the remaining other one to be set free} \end{cases}$$

You could also do a table:

y	x	$f_{y\|x}$

- question: $f_{x|y}(A|B) = f_x(A)$?
  $f_{x|y}(A|C) = f_x(A)$?

Let $i = B, C$ and $\bar{i} = C$ if $i = B$, $\bar{i} = B$ if $i = C$

$$f_{x|y}(A|i) = \frac{f_{y|x}(i|A)\, f_x(A)}{f_y(i)} = \frac{\frac{1}{2}\cdot\frac{1}{3}}{\sum_{k=A,B,C} f_{y|x}(i|k)\cdot f_x(k)}$$

$$= \frac{\frac{1}{6}}{\underbrace{\frac{1}{2}}_{f_{y|x}(i|A)}\cdot\frac{1}{3} + \underbrace{0}_{f_{y|x}(i|i)} + \underbrace{1}_{f_{y|x}(\bar{i}|i)}\cdot\frac{1}{3}}$$

$$= \underline{\frac{1}{3}}$$

$\Rightarrow$ As the value of the poster probability is the same as the prior one $f_x(A)$, the jailer is wrong

$\rightarrow$ no additional information for A from jailer!

III) Check Wikipedia:

- Three prisoners problem
- Monty Hall problem

Problem 10

Consider the joint cumulative distribution:

$$F_{v,\omega}(\bar{v},\bar{\omega}) = Pr(v \le \bar{v}, \omega \le \bar{\omega})$$
$$= Pr(g(x) \le \bar{v}, h(y) \le \bar{\omega})$$

Define the sets

$$A_{\bar{v}} = \{x \in \overline{X} : g(x) \le \bar{v}\}$$
$$A_{\bar{\omega}} = \{y \in \overline{Y} : h(y) \le \bar{\omega}\}$$

Then,

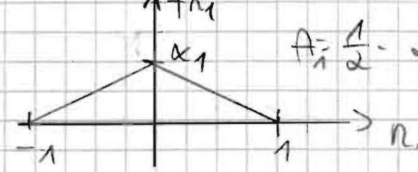
$$F_{v,\omega}(\bar{v},\bar{\omega}) = Pr(x \in A_{\bar{v}}, y \in A_{\bar{\omega}}) \quad \forall \bar{v},\bar{\omega}$$
$$= Pr(x \in A_{\bar{v}})\, Pr(y \in A_{\bar{\omega}}) \quad \text{independence assumption}$$
$$= Pr(g(x) \le \bar{v})\, Pr(h(y) \le \bar{\omega})$$
$$= Pr(v \le \bar{v})\, Pr(\omega \le \bar{\omega})$$
$$= F_v(\bar{v})\, F_\omega(\bar{\omega})$$

q.e.d.

Problem 11

(a) • $\int_{-\infty}^{\infty} f_{n_1}(\bar{n}_1)\,d\bar{n}_1 = \alpha_1 \left[\int_{-1}^{0} (1+\bar{n}_1)\,d\bar{n}_1 + \int_{0}^{1} (1-\bar{n}_1)\,d\bar{n}_1 \right] \overset{!}{=} 1$

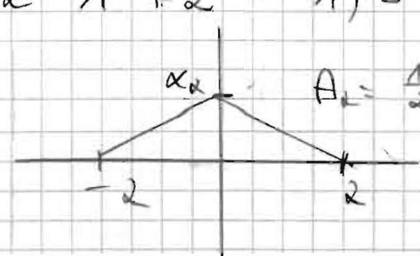
$$= \alpha_1 \left(1 - \tfrac{1}{2} + 1 - \tfrac{1}{2}\right) = 1$$



$\Rightarrow \underline{\underline{\alpha_1 = 1}}$

• $\int_{-\infty}^{\infty} f_{n_2}(\bar{n}_2)\,d\bar{n}_2 = \alpha_2 \left[\int_{-2}^{0} (1+0.5\bar{n}_2)\,d\bar{n}_2 + \int_{0}^{2} (1-0.5\bar{n}_2)\,d\bar{n}_2 \right] \overset{!}{=} 1$

$$= \alpha_2 (2 - 1 + 2 - 1) = 2\alpha_2 = 1$$



$\Rightarrow \underline{\underline{\alpha_2 = \frac{1}{2}}}$

(b) ∘ Without prior information on x, assume all values are equally likely (maximizing the entropy ...)

$f_X(x) = \frac{1}{10}$ for $-5 \leq x \leq 5$

$f_X(x) = 0$ otherwise

Check: $\int_{-\infty}^{\infty} f_X(\bar{x})\, d\bar{x} = 1$.

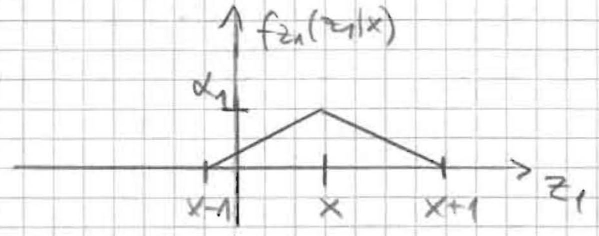
∘ Independent assumption:

$f(z_1, z_2 | x) = f(z_1 | x) \cdot f(z_2 | x)$

∘ Calculate observation likelihood: $f(z_1|x)$, $f(z_2|x)$

$z_1 - x = n_1$, $z_2 - x = n_2$, x is given/fix

$$f_{z_1|x}(z_1|x) = \begin{cases} \alpha_1 (1 - z_1 + x) & \text{for } 0 \leq z_1 - x \leq 1 \\ \alpha_1 (1 + z_1 - x) & \text{for } -1 \leq z_1 - x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$



Similar, for $f_{z_2|x}(z_2|x)$

$$f_{z_2|x}(z_2|x) = \begin{cases} \alpha_2 (1 - \frac{1}{2} z_2 + \frac{1}{2} x) & \text{for } 0 \leq z_2 - x \leq 2 \\ \alpha_2 (1 + \frac{1}{2} z_2 - \frac{1}{2} x) & \text{for } -2 \leq z_2 - x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

∘ Bayes Theorem:

independence assumption

$$f(x | z_1, z_2) = \frac{f(z_1|x)\, f(z_2|x)\, f(x)}{\underbrace{\int_{-5}^{5} f(x) \cdot f(z_1|x) \cdot f(z_2|x)\, dx}_{\substack{\text{normalization} \\ := n(z_1, z_2)}}}$$

$\to$ numerator : $z_1 = 0$, $z_2 = 0$ given

$$num(x) = \frac{1}{10} \cdot f_{z_1|x}(0|x) \cdot f_{z_2|x}(0|x)$$

Consider four different intervals: $[-5,-1]$, $[-1,0]$, $[0,1]$, $[1,5)$

I) $num(x) = 0$ for $-5 \le x \le -1$, $1 \le x \le 5$

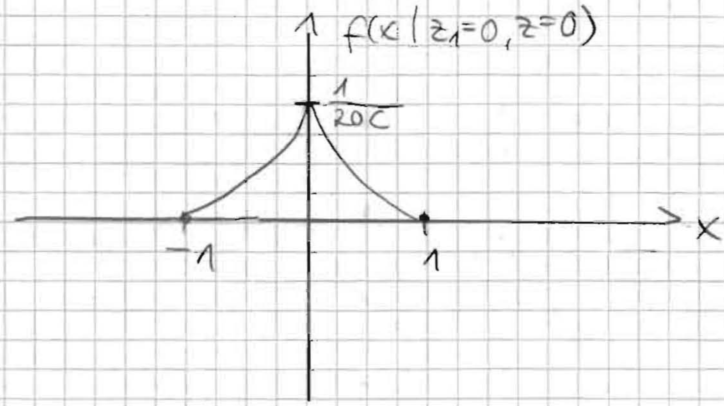
II) $num(x) = \frac{1}{10} \cdot \alpha_1 \cdot (1+x) \cdot \alpha_2 (1+\frac{1}{2}x)$ for $-1 \le x \le 0$

$$= \frac{1}{20} (1+x)(1+\frac{x}{2})$$

III) $num(x) = \frac{1}{10} \alpha_1 (1-x) \alpha_1 (1-\frac{x}{2})$ for $0 \le x \le 1$

$$= \frac{1}{20} (1-x)(1-\frac{x}{2})$$

$$\to f(x|0,0) = \underbrace{\frac{1}{n(0,0)}}_{:=C} \cdot num(x)$$



$\to$ normalize

$$n(0,0) = \int_{-1}^{1} num(x)\,dx = \frac{1}{20}\Big[\int_{-1}^{0} (1+x)(1+\frac{x}{2})\,dx + \int_{0}^{1} (1-x)(1-\frac{x}{2})\,dx$$

$$= \frac{1}{20}\left(\frac{5}{12} + \frac{5}{12}\right) = \frac{1}{24}$$

$$\Rightarrow f(x|0,0) = \begin{cases} 0 & \text{for } -5 \le x \le -1,\ 1 \le x \le 5 \\ \frac{12}{10}(1+x)(1+\frac{x}{2}) & \text{for } -1 \le x \le 0 \\ \frac{12}{10}(1-x)(1-\frac{x}{2}) & \text{for } 0 \le x \le 1 \end{cases}$$

$\Rightarrow$ symmetric to $x=0$, peak at $x=0$, both season "agree"

(c) Similar to (b)

$\rightarrow$ numerator: $z_1 = 0$, $z_2 = 1$ given

$$num(x) = \frac{1}{10} f_{z_1|x}(0|x) f_{z_2|x}(1|x)$$

Consider four different intervals: $[-5,-1]$, $[-1,0]$, $[0,1]$, $[1,5]$

I) $num(x) = 0$ for $-5 \le x \le -1$, $1 \le x \le 5$

II) $num(x) = \frac{1}{10} \alpha_1 (1+x) \alpha_2 (\frac{1}{2} + \frac{1}{2}x)$ for $-1 \le x \le 0$

$$= \frac{1}{40} (1+x)^2$$

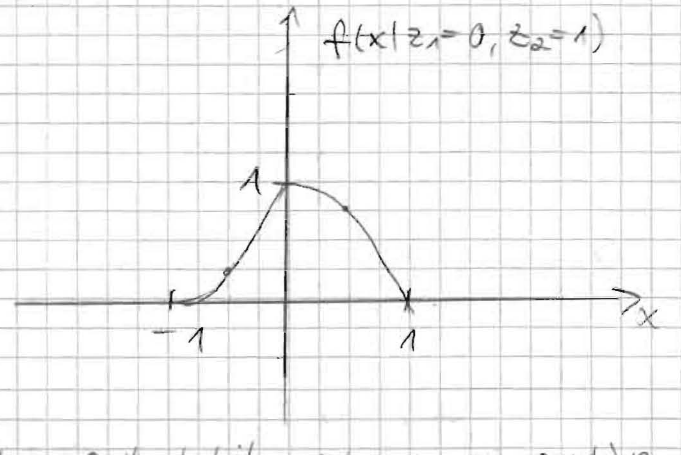
III) $num(x) = \frac{1}{10} \alpha_1 (1-x) \alpha_2 (\frac{1}{2} + \frac{1}{2}x)$ for $0 \le x \le 1$

$$= \frac{1}{40} (1-x^2)$$

$\rightarrow$ normalize

$$n(0,1) = \int_{-1}^{1} num(x)\, dx = \frac{1}{120} + \frac{2}{120} = \frac{1}{40}$$

$$\Rightarrow f(x|0,1) = \begin{cases} 0 & \text{for } -5 \le x \le -1,\ 1 \le x \le 5 \\ (1+x)^2 & \text{for } -1 \le x \le 0 \\ 1-x^2 & \text{for } 0 \le x \le 1 \end{cases}$$



$\Rightarrow$ higher probability values on positive x-axis because of

$z_2 = 1$ measurement

(d) Similar to (b) and (c)

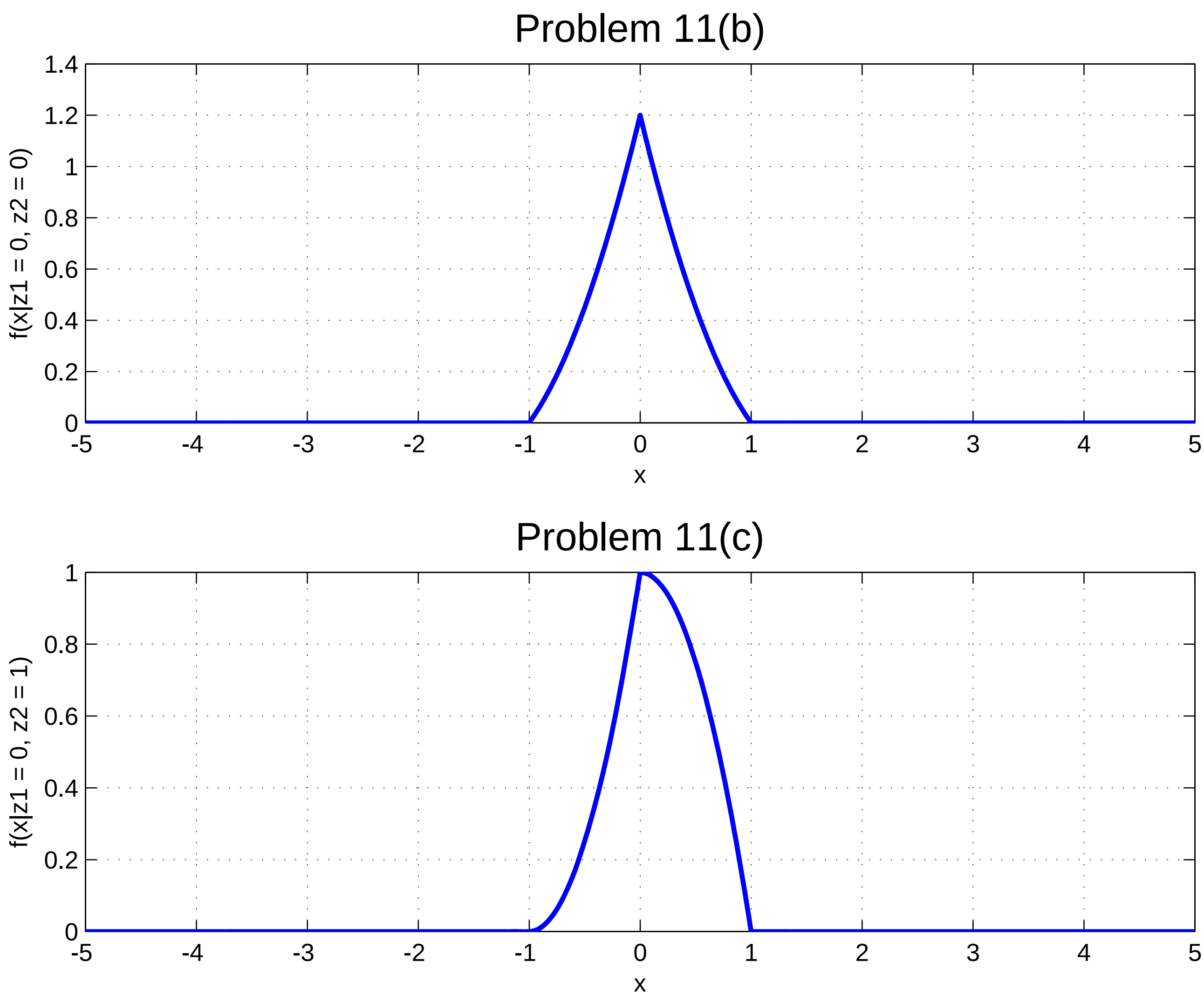
$\rightarrow$ numerator: $z_1 = 0$, $z_2 = 3$ given

$$num(x) = \frac{1}{10} f_{z_1|x}(0|x) \; f_{z_2|x}(3|x)$$

$\Rightarrow$ However, this time the intervals of positive probability of $f_{z_1|x}(0|x)$, $f_{z_2|x}(3|x)$ do not overlap, i.e.

$$num(x) = 0 \qquad \forall x \in [-5,5]$$

i.e. given our noise model for n_1, n_2, there is no chance to measure $z_1 = 0$ and $z_2 = 3$ ...
$f(x | z_1 = 0, z_2 = 3)$ is <u>not defined</u>.


Problem 11(b)
f(x|z1 = 0, z2 = 0)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-5
-4
-3
-2
-1
0
1
2
3
4
5
x
Problem 11(c)
f(x|z1 = 0, z2 = 1)
1
0.8
0.6
0.4
0.2
0
-5
-4
-3
-2
-1
0
1
2
3
4
5
x

Problem 12

* Code attached *

a) - bimodal distribution,
- decay,
- works,
- uniform distribution $\forall\ k = 1, 2, \ldots$

b) - wrong result: estimation vs. real position,
- bimodal - we cannot differentiate
- incorrect assumption
- washes out
- crashes! Think about why... :)

```
% Problem Set2 - Problem 12
%
% *Object on Circle - Recursive Filtering Algorithm*
%
% Recursive Filtering and Estimation
% Spring 2010
%
% --
% ETH Zurich
% Institute for Dynamic Systems and Control
% Angela Schöllig
% aschoellig@ethz.ch
%
% --
% Revision history
% [14.03.10, AS]    first version
%


clear

rand('state',0);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Configuration Constants
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Number of simulation steps
T = 100;

% Number of discrete steps around circle
N = 100;

% Actual probability of going CCW
PROB = 0.55;

% Model of probability of going CCW
PROB_MODEL = PROB;
%PROB_MODEL = 0.45;

% Location of distance sensor, as a multiple of the circle radius.  Can be
% less than 1 (inside circle), but must be positive (WLOG).
SENSE_LOC = 2;

% The sensor error is modeled as additive (a time of flight sensor, for
% example), uniformly distributed around the actual distance.  The units
% are in circle radii.
ERR_SENSE = 0.50;

% Model of what the sensor error is
ERR_SENSE_MODEL = ERR_SENSE;
%ERR_SENSE_MODEL = 0.45;
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Initialization
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% W(k,i) denotes the probability that the object is at location i at time
% k, given all measurements up to, and including, time k.  At time 0, this
% is initialized to 1/N, all positions are equally likely.
W = zeros(T+1,N);
W(0+1,:) = 1/N;

% The intermediate prediction weights, initialize here for completeness.
% We don't keep track of their time history.
predictW = zeros(1,N);

% The initial location of the object, an integer between 0 and N-1.
loc = zeros(T+1,1);
loc(0+1) = round(N/4);

%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%
% Simulation
%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%

for t = 1:T
    %%%%%%%%%%%%%%%%%
    % Simulate System
    %%%%%%%%%%%%%%%%%

    % Process dynamics.  With probability PROB we move CCW, otherwise CW
    if (rand < PROB)
        loc(t+1) = mod(loc(t) + 1,N);
    else
        loc(t+1) = mod(loc(t) - 1,N);
    end

    % The physical location of the object is on the unit circle
    xLoc = cos(2*pi * loc(t+1)/N);
    yLoc = sin(2*pi * loc(t+1)/N);

    % Can calculate the actual distance to the object
    dist = sqrt( (SENSE_LOC - xLoc)^2 + yLoc^2);

    % Corrupt the distance by noise
    dist = dist + ERR_SENSE * 2 * (rand - 0.5);

    %%%%%%%%%%%%%%%%%%
    % Update Estimator
    %%%%%%%%%%%%%%%%%%

    % Prediction Step.  Here we form the intermediate weights which capture
    % the pdf at the current time, but not using the latest measurement.
    for i = 1:N
```

```
        predictW(i) = PROB_MODEL*W(t, 1+mod(i-2,N)) + (1-PROB_MODEL)*W(t, 1+ mod(i,↙
N));
    end

    % Fuse prediction and measurement.  We simply scale the prediction step
    % weight by the conditional probability of the observed measurement
    % at that state.  We then normalize.
    for i = 1:N

        xLocHypo = cos(2*pi * (i-1)/N);
        yLocHypo = sin(2*pi * (i-1)/N);

        distHypo = sqrt( (SENSE_LOC - xLocHypo)^2 + yLocHypo^2);

        if abs(dist-distHypo) < ERR_SENSE_MODEL
            condProb = 1/(2*ERR_SENSE_MODEL);
        else
            condProb = 0;
        end

        W(t+1,i) = condProb * predictW(i);

    end

    % Normalize the weights.  If the normalization is zero, it means that
    % we received an inconsistent measurement.  We can either use the old
    % valid data, re-initialize our estimator, or crash. To be as
    % robust as possible, we simply re-initialize the estimator.
    normConst = sum(W(t+1,:));

    % Uncomment this line if we want to allow the program to crash.
    W(t+1,:) = W(t+1,:)/normConst;    normConst = 1.0;

    if (normConst > 1e-6)
        W(t+1,:) = W(t+1,:)/normConst;
    else
        W(t+1,:) = W(1,:);
    end

end

%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%
% Visualize the results
%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%

figure(1)
xVec = (0:N-1)/N;
yVec = 0:T;
mesh(xVec,yVec,W);
xlabel('POSITION x(k)/N ');
ylabel('TIME STEP k');
view([-30,40]);
hold on
```

```
% actual simulated position
plot3(loc/N,(0:T)',ones(T+1,1)*max(max(W)));
hold off

findfigs
```