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**Quiz 1****October 14th, 2008****Dynamic Programming & Optimal Control (151-0563-00)****Prof. R. D'Andrea**

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# Solutions

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**Exam Duration:** 45 minutes**Number of Problems:** 3**Permitted Aids:** None.Use only the prepared sheets for your solutions.

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**Problem 1****25%**

Consider the system equation

$$\tilde{x}_{k+1} = \tilde{f}_k(\tilde{x}_k, \tilde{x}_{k-2}, \tilde{u}_k),$$

and the cost

$$\tilde{g}_N(\tilde{x}_N) + \sum_{k=0}^{N-1} \tilde{g}_k(\tilde{x}_k, \tilde{x}_{k-2}, \tilde{u}_k).$$

Reformulate this problem in the form of the basic problem that can directly be solved with the Dynamic Programming Algorithm, that is bring the problem in the form

$$x_{k+1} = f_k(x_k, u_k),$$

with the cost

$$g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k).$$

**Solution 1**

Introducing the new state variable

$$x_k = \begin{bmatrix} x_{1,k} \\ x_{2,k} \\ x_{3,k} \end{bmatrix} := \begin{bmatrix} \tilde{x}_k \\ \tilde{x}_{k-1} \\ \tilde{x}_{k-2} \end{bmatrix},$$

and the input variable  $u_k := \tilde{u}_k$ , we can rewrite the system equation

$$x_{k+1} := \begin{bmatrix} \tilde{x}_{k+1} \\ \tilde{x}_k \\ \tilde{x}_{k-1} \end{bmatrix} = \begin{bmatrix} \tilde{f}_k(\tilde{x}_k, \tilde{x}_{k-2}, \tilde{u}_k) \\ \tilde{x}_k \\ \tilde{x}_{k-1} \end{bmatrix} = \begin{bmatrix} \tilde{f}_k(x_{1,k}, x_{3,k}, u_k) \\ x_{1,k} \\ x_{2,k} \end{bmatrix} =: f_k(x_k, u_k). \quad (1)$$

The cost becomes

$$\begin{aligned} \tilde{g}_N(\tilde{x}_N) + \sum_{k=0}^{N-1} \tilde{g}_k(\tilde{x}_k, \tilde{x}_{k-2}, \tilde{u}_k) &= \tilde{g}_N(x_{N,1}) + \sum_{k=0}^{N-1} \tilde{g}_k(x_{1,k}, x_{3,k}, u_k) \\ &= g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k), \end{aligned} \quad (2)$$

where we defined  $g_N(x_N) := \tilde{g}_N(x_{1,N})$  and  $g_k(x_k, u_k) := \tilde{g}_k(x_{1,k}, x_{3,k}, u_k)$ . The system equation (1) and the cost (2) are in the desired form of the basic problem.

**Problem 2****25%**

Consider the cost  $g_k$  at stage  $k$ :

$$g_k(x_k, u_k, w_k) = x_k^2 + 2x_k u_k w_k + 2w_k^2,$$

and the conditional probability distribution for  $w_k$

$$\begin{aligned} P(w_k = 0 | x_k = 0) &= \frac{1}{2} & P(w_k = -1 | x_k = 1) &= \frac{1}{6} \\ P(w_k = 1 | x_k = 0) &= \frac{1}{2} & P(w_k = 0 | x_k = 1) &= \frac{1}{2} \\ & & P(w_k = 1 | x_k = 1) &= \frac{1}{3}. \end{aligned}$$

Given the state  $x_k = 1$  and the input  $u_k = 2$ , compute the expected value

$$E_{w_k}(g_k(x_k, u_k, w_k)).$$

**Solution 2**

In general, the expected value of a function  $f(\cdot)$  of a discrete random variable  $W$  that takes the discrete values  $w_1, w_2, \dots, w_N$  with probability distribution  $P(w_i|x_k)$  conditioned on  $x_k$  is

$$E(f(W)) = \sum_{i=1}^N f(w_i)P(w_i|x_k).$$

For the given problem, since  $x_k = 1$  is given, the expected value of  $g_k$  is

$$\begin{aligned} E(g_k(x_k, u_k, w_k)) &= \sum_{i=1}^3 g_k(x_k = 1, u_k = 2, w_i)P(w_i|x_k = 1) \\ &= \frac{1}{6}g_k(1, 2, -1) + \frac{1}{2}g_k(1, 2, 0) + \frac{1}{3}g_k(1, 2, 1) \\ &= \frac{1}{6} \cdot (1^2 + 2 \cdot 1 \cdot 2 \cdot (-1) + 2 \cdot (-1)^2) \\ &\quad + \frac{1}{2} \cdot (1^2 + 2 \cdot 1 \cdot 2 \cdot 0 + 2 \cdot 0^2) \\ &\quad + \frac{1}{3} \cdot (1^2 + 2 \cdot 1 \cdot 2 \cdot 1 + 2 \cdot 1^2) \\ &= -\frac{1}{6} + \frac{1}{2} + \frac{7}{3} \\ &= \frac{8}{3}. \end{aligned}$$

**Problem 3****50%**

Consider the dynamic system

$$x_{k+1} = x_k + u_k, \quad x_k, u_k \in \mathbb{R},$$

with initial state  $x_0$ . The cost function, to be minimized, is given by

$$x_2^2 + u_0^2 + u_1^2.$$

Apply the Dynamic Programming algorithm to find the optimal control policy  $u_k^* = \mu_k^*(x_k)$ ,  $k = 0, 1$ , for the following two cases:

- a) no constraints on  $u_k$ .
- b)  $u_k$  can only take the values 1 and  $-1$ .

**Solution 3**

The optimal control problem is considered over a time horizon  $N = 2$  and the cost, to be minimized, is defined by

$$g_2(x_2) = x_2^2 \quad \text{and} \quad g_k(x_k, u_k, w_k) = u_k^2, \quad k = 0, 1.$$

a) With  $u_k \in \mathbb{R}$  and no further constraints, the DP algorithm proceeds as follows:

**2nd stage:**

$$J_2(x_2) = x_2^2$$

**1st stage:**

$$J_1(x_1) = \min_{u_1} \left[ u_1^2 + J_2(x_2) \right] = \min_{u_1} \underbrace{\left[ u_1^2 + (x_1 + u_1)^2 \right]}_{L_1(x_1, u_1)}$$

Since  $u_1$  is continuous, the minimizing input is found by

$$\frac{\partial L_1}{\partial u_1} = 2u_1 + 2(x_1 + u_1) \stackrel{!}{=} 0 \quad \Leftrightarrow \quad u_1 = -\frac{1}{2}x_1.$$

The sufficient condition  $\frac{\partial^2 L_1}{\partial u_1^2} > 0$  is satisfied. Therefore,

$$\begin{aligned} \Rightarrow \mu_1^*(x_1) &= u_1^* = -\frac{1}{2}x_1 \quad \forall x_1 \in \mathbb{R}, \\ \Rightarrow J_1(x_1) &= \frac{1}{2}x_1^2. \end{aligned}$$

**0th stage:**

$$J_0(x_0) = \min_{u_0} \left[ u_0^2 + J_1(x_1) \right] = \min_{u_0} \underbrace{\left[ u_0^2 + \frac{1}{2}(x_0 + u_0)^2 \right]}_{L_0(x_0, u_0)}$$

Again, since  $u_0$  is continuous, the minimizing input is found by

$$\frac{\partial L_0}{\partial u_0} = 2u_0 + (x_0 + u_0) \stackrel{!}{=} 0 \quad \Leftrightarrow \quad u_0 = -\frac{1}{3}x_0.$$

The sufficient condition  $\frac{\partial^2 L_0}{\partial u_0^2} > 0$  is satisfied. Therefore,

$$\begin{aligned} \Rightarrow \mu_0^*(x_0) &= u_0^* = -\frac{1}{3}x_0 \quad \forall x_0 \in \mathbb{R}, \\ \Rightarrow J_0(x_0) &= \frac{1}{3}x_0^2. \end{aligned}$$

b) For the constrained set  $u_k \in \{-1, 1\}$ , the DP algorithm results in:

**2nd stage:**

$$J_2(x_2) = x_2^2$$

**1st stage:**

$$\begin{aligned} J_1(x_1) &= \min_{u_1} \left[ u_1^2 + (x_1 + u_1)^2 \right] = 1 + \min_{u_1} \left[ (x_1 + u_1)^2 \right] \\ &= 2 + x_1^2 + 2 \cdot \min_{u_1} [x_1 u_1] \end{aligned}$$

Depending on the sign of  $x_1$ ,

$$\operatorname{sgn}(x_1) = \begin{cases} 1 & \text{for } x_1 \geq 0 \\ -1 & \text{for } x_1 < 0, \end{cases}$$

the minimizing input  $u_1^*$  is found as

$$\begin{aligned} \Rightarrow \mu_1^*(x_1) &= u_1^* = -\operatorname{sgn}(x_1) \quad \forall x_1 \in \mathbb{R}, \\ \Rightarrow J_1(x_1) &= 1 + (1 - |x_1|)^2 = 2 + x_1^2 - 2|x_1|. \end{aligned}$$

**0th stage:**

$$\begin{aligned} J_0(x_0) &= \min_{u_0} \left[ u_0^2 + J_1(x_1) \right] = \min_{u_0} \left[ u_0^2 + 1 + (1 - |x_0 + u_0|)^2 \right] \\ &= 2 + \min_{u_0} \left[ (1 - |x_0 + u_0|)^2 \right] \\ &= 4 + x_0^2 + 2 \cdot \min_{u_0} [x_0 u_0 - |x_0 + u_0|] \end{aligned}$$

Distinguishing the cases  $x_0 < -1$ ,  $-1 \leq x_0 \leq 1$  and  $x_0 > 1$ , yields

$$\mu_0^*(x_0) = u_0^* = \begin{cases} 1 & \text{for } x_0 < -1 \\ \pm 1 & \text{for } -1 \leq x_0 \leq 1 \\ -1 & \text{for } x_0 > 1. \end{cases}$$

In short, *one* optimal solution is

$$\begin{aligned} \Rightarrow \mu_0^*(x_0) &= u_0^* = -\operatorname{sgn}(x_0) \quad \forall x_0 \in \mathbb{R}, \\ \Rightarrow J_0(x_0) &= \begin{cases} 2 + x_0^2 & \text{for } -1 \leq x_0 \leq 1 \\ 6 + x_0^2 - 4|x_0| & \text{otherwise.} \end{cases} \end{aligned}$$