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Dynamic Programming and Optimal Control

Fall 2009

Problem Set:
Deterministic Continuous-Time Optimal Control

Notes:

- Problems marked with BERTSEKAS are taken from the book *Dynamic Programming and Optimal Control by Dimitri P. Bertsekas, Vol. I, 3rd edition, 2005, 558 pages, hardcover.*
- The solutions were derived by the teaching assistants in the previous class. Please report any error that you may find to `strimpe@ethz.ch` or `aschoellig@ethz.ch`.

Problem Set 3

Problem 1 (LQR)

In the LQR problem discussed in class we assumed that

1. the optimal cost to go is of the form $x^T K(t)x$,

2. the matrix $K(t)$ is symmetric.

To rigorously show that (1) is true a-priori is not trivial, and is beyond the scope of the class. We will tackle (2): prove that if the optimal cost to go is of the form $x^T K(t)x$, then one can assume, without loss of generality, that $K(t)$ is symmetric.

Problem 2 (BERTSEKAS, p. 143, exercise 3.2)

A young investor has earned in the stock market a large amount of money S and plans to spend it so as to maximize his enjoyment through the rest of his life without working. He estimates that he will live exactly T more years and that his capital $x(t)$ should be reduced to zero at time T, i.e., $x(T) = 0$. Also he models the evolution of his capital by the differential equation

$$\frac{dx(t)}{dt} = \alpha x(t) - u(t),$$

where $x(0) = S$ is his initial capital, $\alpha > 0$ is a given interest rate, and $u(t) \geq 0$ is his rate of expenditure. The total enjoyment he will obtain is given by

$$\int_0^T e^{-\beta t} \sqrt{u(t)}\, dt.$$

Here β is some positive scalar, which serves to discount future enjoyment. Find the optimal $\big\{u(t) \mid t \in [0, T]\big\}$.

Problem 3 (Isoperimetric Problem, BERTSEKAS, p. 144, exercise 3.5)

Analyze the problem of finding a curve $\big\{x(t) \mid t \in [0, T]\big\}$ that maximizes the area under x,

$$\int_0^T x(t) dt,$$

subject to the constraints

$$x(0) = a, \quad x(T) = b, \quad \int_0^T \sqrt{1 + \big(\dot{x}(t)\big)^2}\, dt = L,$$

where a, b, and L are given positive scalars. The last constraint is known as an isoperimetric constraint; it requires that the length of the curve be L. *Hint*: Introduce the system $\dot{x}_1 = u$, $\dot{x}_2 = \sqrt{1 + u^2}$, and view the problem as a fixed terminal state problem. Show that the sine of the optimal $u^*(t)$ depends linearly on t.[1] Under some assumptions on a, b and L, the optimal curve is a circular arc.

[1]This is partly misleading. It should read: Show that the sine of the slope angle ϕ, defined by $\tan(\phi) = \frac{dx}{dt}$, is affine linear in t, i.e. $ct + d$ with constants c and d.

Problem 4 (BERTSEKAS, p. 145, exercise 3.7)

A boat moves with constant unit velocity in a stream moving at constant velocity s. The problem is to find the steering angle $u(t)$, $0 \leq t \leq T$, which minimizes the time T required for the boat to move between the point $(0, 0)$ to a given point (a, b). The equations of motion are

$$\dot{x}_1(t) = s + \cos u(t), \qquad \dot{x}_2(t) = \sin u(t),$$

where $x_1(t)$ and $x_2(t)$ are the positions of the boat parallel and perpendicular to the stream velocity, respectively. Show that the optimal solution is to steer at a constant angle.

Sample Solutions

Problem 1 (Solution)

Consider a solution of the form

$$V(t,x) = x^T K(t) x = x^T K x \qquad \text{(drop argument for convenience)}$$

with a general square matrix $K \in \mathbb{R}^{n \times n}$.

- Decompose K into symmetric and skew-symmetric parts, K_s and $K_{\bar{s}}$, respectively,

  $$K = \underbrace{\frac{1}{2}K + \frac{1}{2}K^T}_{=:K_s} + \underbrace{\frac{1}{2}K - \frac{1}{2}K^T}_{=:K_{\bar{s}}},$$

  where $K_s^T = K_s$ and $K_{\bar{s}}^T = -K_{\bar{s}}$.

- For a skew-symmetric matrix $K_{\bar{s}}$, it holds

  $$\begin{aligned} x^T K_{\bar{s}} x &= (x^T K_{\bar{s}} x)^T \qquad (x^T K_{\bar{s}} x \text{ is a scalar}) \\ &= x^T K_{\bar{s}}^T x \\ &= -x^T K_{\bar{s}} x \\ \Leftrightarrow x^T K_{\bar{s}} x &= -x^T K_{\bar{s}} x \quad \Rightarrow \quad x^T K_{\bar{s}} x = 0. \end{aligned}$$

- We write for $V(t,x)$,

  $$V(t,x) = x^T K x = x^T (K_s + K_{\bar{s}}) x = x^T K_s x + \underbrace{x^T K_{\bar{s}} x}_{0} = x^T K_s x.$$

  Therefore, without loss of generality, one can assume $V(t,x) = x^T K x$ with K symmetric.

Problem 2 (Solution)

- system:

  $$\frac{dx}{dt} = \alpha x - u, \quad x(T) = 0, \quad x(0) = S, \quad \alpha > 0$$

- "control" $\rightarrow$ expenditure $u(t) \geq 0 \quad \forall t$
- total gain $\rightarrow$ total enjoyment[2]

  $$\int_0^T e^{-\beta t} \sqrt{u(t)} dt \quad , \quad \beta > 0$$

Apply ***Minimum Principle***

- Hamiltonian:

  $$\begin{aligned} H(x,u,p) &= g(x,u) + p^T f(x,u) \\ H(x,u,p) &= -e^{-\beta t}\sqrt{u} + p(\alpha x - u) \end{aligned}$$

[2]Here, the cost function $g(\cdot)$ explicitly depends on t. Refer to Sec. 3.4.4 of the class textbook for time-varying cost.

- Adjoint equation:

$$\dot{p} = -\nabla_x H(x^*, u^*, p) = -\alpha p$$

$$\Rightarrow \quad \underline{p(t) = c_1 e^{-\alpha t}}$$

- Find minimizing u^*:

$$\begin{aligned} u^* &= \arg \min_{u \geq 0} \; H(x^*, u, p) \\ &= \arg \min_{u \geq 0} \left[-e^{-\beta t} \sqrt{u} + p(\alpha x^* - u) \right] \end{aligned}$$

necessary condition: 1^{st} derivative $= 0$:

$$\frac{d}{du} H = -e^{-\beta t} \frac{1}{2} u^{-\frac{1}{2}} - p = 0$$

$$\Rightarrow \quad u^*(t) = \frac{1}{4p^2} e^{-2\beta t}$$

sufficient condition: 2^{nd} derivative $\neq 0$

$$\frac{d^2}{du^2} H = e^{-\beta t} \frac{1}{2} \cdot \frac{1}{2} u^{-\frac{3}{2}} = \frac{1}{4} e^{-\beta t} \frac{1}{\sqrt{u^3}} > 0 \quad \forall t, u$$

$$\Rightarrow \quad u^*(t) = \frac{1}{4p^2} e^{-2\beta t} \text{ is a minimum.}$$

- Thus, minimizing u^* is

$$u^*(t) = \frac{1}{4{c_1}^2} e^{(2\alpha - 2\beta)t}.$$

We still need to determine c_1, which will be done in the following.

- System equation with optimal u^*:

$$\dot{x} = \alpha x - \frac{1}{4{c_1}^2} e^{(2\alpha - 2\beta)t} \tag{1}$$

Equation (1) is a linear ODE. Its solution consists of the homogeneous solution $x_h(t)$ and a particular solution $x_p(t)$: $x(t) = x_h(t) + x_p(t)$.

Homogeneous solution:

$$x_h(t) = c_2 e^{\alpha t} \quad , \quad c_2 = \text{constant}$$

Particular solution:
Case: $\alpha \neq 2\beta$:

Guessing

$$x_p(t) = c_3 e^{(2\alpha - 2\beta)t}$$

and plugging it into the ODE, yields

$$c_3 (2\alpha - 2\beta) e^{(2\alpha - 2\beta)t} = \alpha c_3 e^{(2\alpha - 2\beta)t} - \frac{1}{4{c_1}^2} e^{(2\alpha - 2\beta)t}.$$

Thus,

$$c_3 = -\frac{1}{4{c_1}^2(\alpha - 2\beta)}$$

$$\Rightarrow \quad x_p(t) = -\frac{1}{4{c_1}^2(\alpha - 2\beta)} e^{(2\alpha-2\beta)t} \text{ is a particular solution.}$$

Thus, the general solution is

$$x(t) = x_h(t) + x_p(t) = c_2 e^{\alpha t} - \frac{1}{4{c_1}^2(\alpha - 2\beta)} e^{(2\alpha-2\beta)t}.$$

Determine c_1 and c_2 from $x(0) = S$ and $x(T) = 0$:

$$\begin{aligned} \frac{1}{4{c_1}^2} &= \frac{-S(\alpha - 2\beta)}{1 - e^{(\alpha-2\beta)T}} \\ c_2 &= \frac{-Se^{(\alpha-2\beta)T}}{1 - e^{(\alpha-2\beta)T}}. \end{aligned}$$

<u>Case: $\alpha = 2\beta$</u>:

ODE:

$$\dot{x} = \alpha x - \frac{1}{4{c_1}^2} e^{\alpha t}$$

Guessing

$$x_p(t) = c_4 t e^{\alpha t}$$

and plugging it into ODE, yields

$$c_4 e^{\alpha t} + c_4 \alpha t e^{\alpha t} = c_4 \alpha t e^{\alpha t} - \frac{1}{4{c_1}^2} e^{\alpha t}.$$

Thus,

$$c_4 = -\frac{1}{4{c_1}^2}.$$

General solution:

$$x(t) = x_h(t) + x_p(t) = c_2 e^{\alpha t} - \frac{1}{4{c_1}^2} t e^{\alpha t}.$$

Determine c_1 and c_2 from $x(0) = S$ and $x(T) = 0$:

$$\begin{aligned} \frac{1}{4{c_1}^2} &= \frac{S}{T} \\ c_2 &= S. \end{aligned}$$

Therefore, the resulting optimal control u^* and optimal state trajectory x^* are:

$$\underline{\alpha \neq 2\beta} : x^*(t) = \frac{-Se^{(\alpha-2\beta)T}}{1 - e^{(\alpha-2\beta)T}} e^{\alpha t} + \frac{S}{1 - e^{(\alpha-2\beta)T}} e^{(2\alpha-2\beta)t}$$

$$u^*(t) = \frac{S(2\beta - \alpha)}{1 - e^{(\alpha-2\beta)T}} e^{(2\alpha-2\beta)t}$$

$$\underline{\alpha = 2\beta} : x^*(t) = Se^{\alpha t} - \frac{S}{T} t e^{\alpha t} = S\left(1 - \frac{t}{T}\right) e^{\alpha t}$$

$$u^*(t) = \frac{S}{T} e^{(2\alpha-2\beta)t} = \frac{S}{T} e^{\alpha t}$$

Problem 3 (Solution)

- system:

$$\dot{x}_1(t) = \dot{x}(t) = u(t)$$
$$\dot{x}_2(t) = \sqrt{1 + \big(u(t)\big)^2}$$

$$x_1(0) = a \quad , \quad x_1(T) = b$$
$$x_2(0) = 0 \quad , \quad x_2(T) = L$$

$$\text{since} \quad \int_0^T \sqrt{1+u^2}dt = \int_0^T \dot{x}_2 dt = x_2|_0^T = x_2(T) - x_2(0) = L$$

- maximize

$$\int_0^T x_1 dt = \int_0^T x dt \quad \Leftrightarrow \quad \min \int_0^T -x_1 dt$$

Apply *Minimum Principle*

- Hamiltonian:

$$H = g + p^T f = (-x) + \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} u \\ \sqrt{1+u^2} \end{bmatrix}$$
$$H = -x_1 + p_1 u + p_2 \sqrt{1+u^2}$$

- Adjoint equation:

$$\dot{p} = -\nabla_x H = -\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad p_1(t) = t - c_1 \quad , \quad c_1 = \text{ constant}$$
$$p_2(t) = c_2 \quad , \quad c_2 = \text{ constant}$$

- Optimal control:

$$u^* = \arg\min_u H = \arg\min_u \underbrace{\left(-x_1^* + p_1 u + p_2 \sqrt{1+u^2}\right)}_{(*)}$$

Differentiate $(*)$ with respect to u:

$$\frac{d}{du}: \quad p_1 + p_2 \frac{u}{\sqrt{1+u^2}} = 0$$
$$\Leftrightarrow \quad \frac{u}{\sqrt{1+u^2}} = \frac{-p_1}{p_2} = \frac{c_1 - t}{c_2} \tag{2}$$

Second derivative of $(*)$:

$$\frac{d^2}{du^2}: \quad \frac{p_2}{\sqrt{1+u^2}} \left(\frac{1}{1+u^2}\right) > 0 \quad \text{(since } p_2 > 0 \text{ which will be seen later)}$$

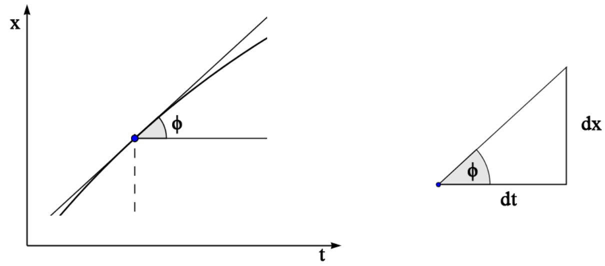
- We have from (2),

$$\frac{\dot{x}^*}{\sqrt{1+\dot{x}^{*2}}} = \frac{c_1 - t}{c_2}, \tag{3}$$

which has to be solved by the wanted curve $x^*(t)$. We will show next, that (3) is solved by a circular arc.

We consider a graphical solution:[3]

- Let ϕ be the *slope angle*, i.e. the angle defined by $\tan\bigl(\phi(t)\bigr) = \dot{x}(t) = \frac{dx}{dt}$.



- Note that

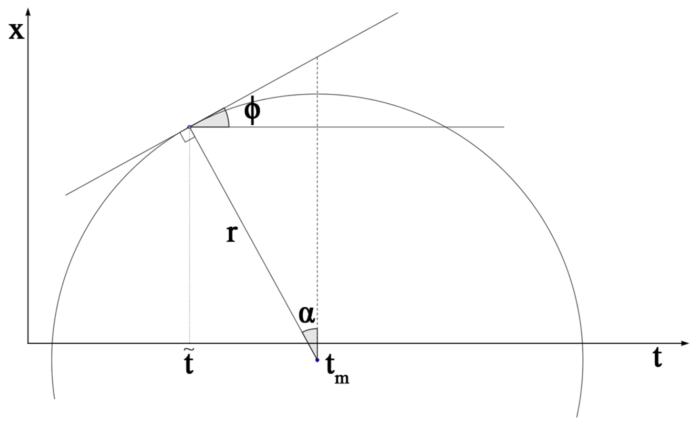
$$\sin\phi = \frac{dx}{\sqrt{dt^2+dx^2}} = \frac{\frac{dx}{dt}}{\sqrt{1+\frac{dx^2}{dt^2}}} = \frac{\dot{x}}{\sqrt{1+\dot{x}^2}}.$$

- With (3), we have

$$\sin\bigl(\phi(t)\bigr) = \frac{c_1 - t}{c_2}, \tag{4}$$

that is, the sine of ϕ is affine linear in t.

- The condition (4) is satisfied by a circle, which can be seen from the following drawing:



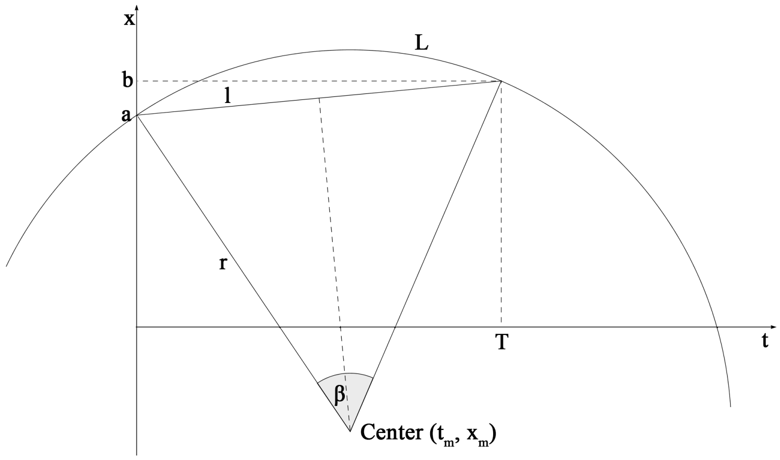
and by noting that $\alpha = \phi$ and

$$\sin(\alpha) = \frac{t_m - \widetilde{t}}{r},$$

where $\widetilde{t}$ is the parameter that changes as one moves along the curve.

[3] Alternatively, it can be shown that the circle equation $(x - x_m)^2 + (t - t_m)^2 = r^2$ solves (3).

Now that we have shown that the problem is solved by a circular arc, we can derive the parameters defining the circle from geometric reasoning. From the following drawing, we get:



- Arc length: $L = \beta r$
- Length l of secant from $(0, \alpha)$ to (T, b): $l = \sqrt{(b-a)^2 + T^2}$
- For β, it holds

  $$\sin\left(\frac{\beta}{2}\right) = \frac{\frac{l}{2}}{r}$$

  with $\beta = \frac{L}{r}$ can solve this for r (e.g. numerically).
- The missing parameters t_m, x_m in the circle equation $(x - x_m)^2 + (t - t_m)^2 = r^2$ can be obtained by plugging in the points $(0, a)$, (T, b):

  $$\begin{aligned} x(0) &= \sqrt{r^2 - t_m^2} + x_m = a \\ x(T) &= \sqrt{r^2 - (T - t_m)^2} + x_m = b, \end{aligned}$$

  which can be solved for t_m, x_m.

Such a circular arc does not exist if L is either too small or too large.

Problem 4 (Solution)

- system:

  $$\begin{aligned} \dot{x}_1(t) &= s + \cos\big(u(t)\big) \\ \dot{x}_2(t) &= \sin\big(u(t)\big) \\ 0 \leq t &\leq T \end{aligned}$$

- minimize the time T to go from $\big[x_1(0), x_2(0)\big] = \big[0, 0\big]$ to $\big[x_1(T), x_2(T)\big] = \big[a, b\big]$

  $$\begin{aligned} &\rightarrow \text{cost } = \int_0^T 1 dt = T \\ &\rightarrow g(x, u) = 1 \end{aligned}$$

Apply *Minimum Principle*

- Hamiltonian:

$$H = 1 + p^T f(x,u)$$
$$H = 1 + p_1\big(s + \cos(u)\big) + p_2\big(\sin(u)\big)$$

- Adjoint equation:

$$\dot{p}(t) = -\nabla_x H = -\begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \end{bmatrix} = 0$$

$$\Rightarrow \quad p_1(t) = c_1 = \text{ const}$$
$$p_2(t) = c_2 = \text{ const}$$

- Optimal $u^*(t)$:

$$u^* = \arg\min_{u\in U} H = \arg\min_{u}\Big(1 + p_1\big(s + \cos(u)\big) + p_2\big(\sin(u)\big)\Big)$$

Differentiate with respect to u and set to 0:[4]

$$\frac{d}{du} : -p_1 \sin(u) + p_2 \cos(u) = 0$$
$$\Rightarrow \quad u = \tan^{-1}\left(\frac{c_2}{c_1}\right) =: \Theta = \text{const}$$

- To get the optimal angle, we plug in $u = \Theta$ into the system equation and solve the ODE:

$$\dot{x}_1 = s + \cos(\Theta)$$
$$\dot{x}_2 = \sin(\Theta)$$

$$\rightarrow x_1(t) = \big(s + \cos(\Theta)\big)t + c_3$$
$$x_2(t) = \sin(\Theta)t + c_4$$

with constants c_3, $c_4 \in \mathbb{R}$.

- Plug in initial and terminal values

$$x_1(0) = c_3 = 0 \quad \Rightarrow \quad c_3 = 0$$
$$x_2(0) = c_4 = 0 \quad \Rightarrow \quad c_4 = 0$$

$$x_1(T) = \big(s + \cos(\Theta)\big)T = a$$
$$x_2(T) = \sin(\Theta)T = b$$

The last to equations can be solved for the unknowns Θ and T for given a, b, s.

[4] Note that we would have to check that this is indeed a minimum (e.g. by checking 2^{nd} derivative). Here, however, we only want to show that the minimum, which we know that it exists from the problem description, is constant.