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Dynamic Programming and Optimal Control

Fall 2009

Problem Set:
Infinite Horizon Problems, Value Iteration, Policy Iteration

Notes:

- Problems marked with BERTSEKAS are taken from the book *Dynamic Programming and Optimal Control by Dimitri P. Bertsekas, Vol. I, 3rd edition, 2005, 558 pages, hardcover.*
- The solutions were derived by the teaching assistants. Please report any error that you may find to `strimpe@ethz.ch` or `aschoellig@ethz.ch`.
- The solutions in this document are handwritten. A typed version will be available in future.

Problem Set

Problem 1 (BERTSEKAS, p. 445, exercise 7.1)

Problem 2 (BERTSEKAS, p. 446, exercise 7.3)

Problem 3 (BERTSEKAS, p. 448, exercise 7.12)

Problem 4 (BERTSEKAS, p. 60, exercise 1.23)

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Exercise 7.1

- First, recist the tennis rules... ☺

For a stochastic shortest path problem, we need to define
(a) set of states
(b) transition probabilities
(c) control input
(d) cost per stage / step

(a) state space

- $x_k = (u_k, v_k, w_k) \in \{0,1,2,3\} \times \{0,1,2,3\} \times \{1,2\}$

with u_k are points of player 1
v_k " " " " 2
w_k denotes first or second serve of player 1

Note $u_k = 1$ corresponds to score of player 1 is 15
$u_k = 2$ " " " " " 1 " 30
$u_k = 3$ " " " " " 1 " 40.

Note $(3,3,w_k)$ corresponds also to "deuce" situation.
$(3,2,w_k)$ corresponds also to "advantage" for player 1.
$(2,3,w_k)$ corresponds also to "advantage" for player 2.

- We have <u>two</u> termination states
$x_k = g_1$ player 1 wins the game corresponds to $(4,2,w_k)$
$x_k = g_2$ " 2 " " " " " $(2,4,w_k)$

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(c) control input

- $u_k \in \{F, S\}$ Fast serve or Slow serve

(b) transition probabilities

Let's consider which transitions are possible and assign probabilities p and q - not yet specifying which input was applied.

conditional probabilities

- $(u_k, v_k, 1) \xrightarrow{qp} (u_k+1, v_k, 1)$ for $u_k \neq 3$, $v_k \neq 3$
  $\xrightarrow{(1-q)p} (u_k, v_k+1, 1)$
  $\xrightarrow{(1-p)} (u_k, v_k, 2)$
- $(u_k, v_k, 2) \xrightarrow{qp} (u_k+1, v_k, 1)$ for $u_k \neq 3$, $v_k \neq 3$
  $\xrightarrow{(1-q)p+(1-p)} (u_k, v_k+1, 1)$
- $(3, v_k, 1) \xrightarrow{qp} g_1$ for $v_k \neq 3$
  $\xrightarrow{(1-q)p} (3, v_k+1, 1)$
  $\xrightarrow{(1-p)} (3, v_k, 2)$
- $(3, v_k, 2) \xrightarrow{qp} g_1$ for $v_k \neq 3$
  $\xrightarrow{(1-q)p+(1-p)} (3, v_k+1, 1)$
- $(u_k, 3, 1) \xrightarrow{qp} (u_k+1, 3, 1)$ for $u_k \neq 3$
  $\xrightarrow{(1-q)p} g_2$
  $\xrightarrow{(1-p)} (u_k, 3, 2)$
- $(u_k, 3, 2) \xrightarrow{qp} (u_k+1, 3, 1)$ for $u_k \neq 3$
  $\xrightarrow{(1-q)p+(1-p)} g_2$
- $(3, 3, 1) \xrightarrow{qp} (3, 2, 1)$ "advantage"
  $\xrightarrow{(1-q)p} (2, 3, 1)$ "
  $\xrightarrow{(1-p)} (3, 3, 2)$
- $(3, 3, 2) \xrightarrow{qp} (3, 2, 1)$ "
  $\xrightarrow{(1-q)p+(1-p)} (2, 3, 1)$

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(d) cost per stage

- The objective is to maximize the probability of winning the game, i.e. we do not have stage costs but rather can express this goal by defining the optimal cost

$$J^*(g_1) = 1$$
$$J^*(g_2) = 0$$

In all equations, we always can replace "min" by "max".

- ⇒ Assumption 7.2.1. is satisfied since from each state defined above there is a positive probability to end the game.

⇒ Bellman's Equation:

$$J^*(3,3,2) = \max_{u \in \{S,F\}} \left(q_u p_u J^*(3,2,1) + (1 - q_u p_u) J^*(2,3,1) \right)$$

$$J^*(3,3,1) = \max_{u \in \{S,F\}} \left(q_u p_u J^*(3,2,1) + (1-q_u) p_u J^*(2,3,1) + (1-p_u) J^*(3,3,2) \right)$$

$$J^*(u_k,3,2) = \max_{u \in \{S,F\}} \left(q_u p_u J^*(u_k+1,3,1) \right)$$

for $u_k \neq 3$

$$J^*(u_k,3,1) = \max_{u \in \{S,F\}} \left(q_u p_u J^*(u_k+1,3,1) + (1-p_u) J^*(u_k,3,2) \right)$$

for $u_k \neq 3$

$$J^*(3,v_k,2) = \max_{u \in \{S,F\}} \left(q_u p_u + (1 - q_u p_u) J^*(3,v_k+1,1) \right)$$

for $v_k \neq 3$

$$J^*(3,v_k,1) = \max_{u \in \{S,F\}} \left(q_u p_u + (1-q_u) p_u J^*(3,v_k+1,1) + (1-p_u) J^*(3,v_k,2) \right)$$

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for $v_k \neq 3$

$$J^*(u_k, v_k, 2) = \max_{u \in \{S, F\}} \left(q_u p_u J^*(u_{k+1}, v_k, 1) + (1 - q_u p_u) J^*(u_k, v_{k+1}, 1) \right)$$

for $u_k \neq 3, v_k \neq 3$

$$J^*(u_k, v_k, 1) = \max_{u \in \{S, F\}} \left(q_u p_u J^*(u_{k+1}, v_k, 1) + (1 - q_u p_u) J^*(u_k, v_{k+1}, 1) + (1 - p_u) J^*(u_k, v_k, 2) \right)$$

for $u_k \neq 3, v_k \neq 3$

b) see Code

Plot shows three different strategies

(1) always slow serve
(2) slow serve in first serve,
fast " " 2nd "
(3) always fast serve

```
% Solution to Exercise 7.1 b)
% in Bertsekas - "Dynamic Programming and Optimal Control" Vol. 1 p. 445
%
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% --
% Revision history
% [12.11.09, AS]    first version
%
%


% clear workspace and command window
clear;
clc;

%% PARAMETERS
% Landing probability
p(2) = 0.95; % Slow serve

% Winning probability
q(1) = 0.6; % Fast serve
q(2) = 0.4; % Slow serve

% Define value iteration error bound
err = 1e-100;


% Specifiy if you want to test one value for p(1) - landing probability for
% fast serve: use test = 1, or a row of probabilities: use test = 2
test =1;

if test ==1
    % Define p(1)
    p(1) = 0.2;
    % Number of runs
    mend = 1;
else
    % Define increment for p(1)
    incr = 0.01;
    % Number of runs
    mend = ((0.95-incr)/incr+2);
end



%% VALUE ITERATION
for m=1:mend
        %% PARAMETER
        % Landing probability
        if test == 2
```
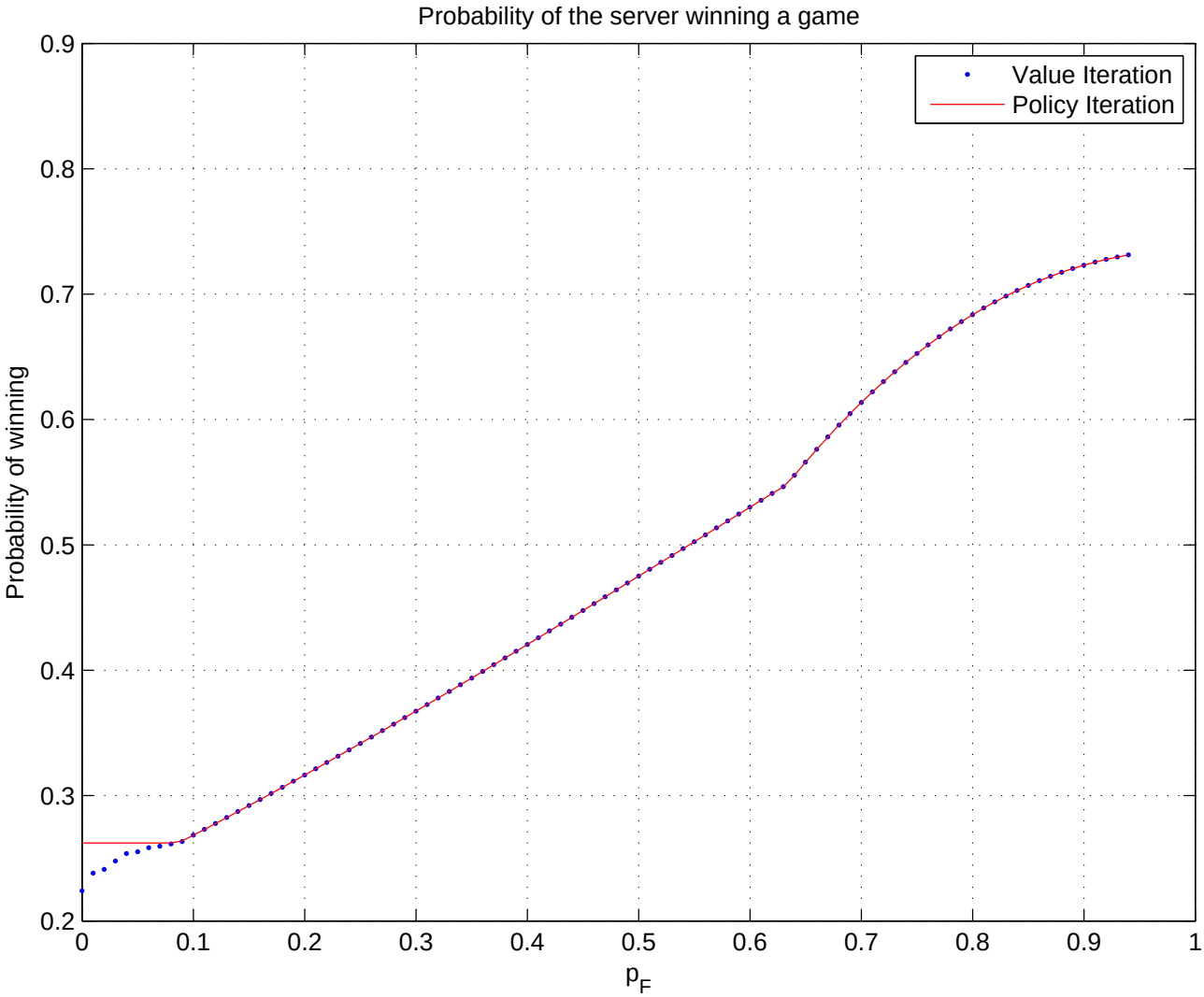

```
            p(1) = (m-1)*incr; % Fast serve, check for 0...0.9
        end


        %% INITIALIZE PROBLEM
        % Our state space is S={0,1,2,3}x{0,1,2,3}x{1,2},
        % i.e. x_k = [score player 1, score player 2, serve]
        % ATTENTION: indices are shifted in code

        % Set the initial costs to 0, although any value will do (except for the
        % termination cost, which must be equal to 0).
        J = (p(1)+0.1).*ones(4, 4, 2);

        % Initialize the optimal control policy: 1 represents Fast serve, 2
        % represents Slow serve
        FVal = ones(4, 4, 2);

        %% POLICY ITERATION

        % Initialize cost-to-go
        costToGo = zeros(4, 4, 2);

        iter = 0;
        while (1)
            iter = iter+1;
            if (mod(iter,1e3)==0)
                disp(['Value Iteration Number ',num2str(iter)]);
            end

            % Update the value
            for i = 1:3
                    [costToGo(4,i,1),FVal(4,i,1)] = max(q.*p + (1-q).*p.*J(4,i+1,1)+↙
(1-p).*J(4,i,2));
                    [costToGo(4,i,2),FVal(4,i,2)] = max(q.*p + (1-q.*p).*J(4,i+1,1));
                    [costToGo(i,4,1),FVal(i,4,1)] = max(q.*p.*J(i+1,4,1) + (1-p).*J↙
(i,4,2));
                    [costToGo(i,4,2),FVal(i,4,2)] = max(q.*p.*J(i+1,4,1));
                for j = 1:3
                    % Maximize over input 1 and 2
                    [costToGo(i,j,1),FVal(i,j,1)] = max(q.*p.*J(i+1,j,1) + (1-q).*p.↙
*J(i,j+1,1)+(1-p).*J(i,j,2));
                    [costToGo(i,j,2),FVal(i,j,2)] = max(q.*p.*J(i+1,j,1) + (1-q.*p).↙
*J(i,j+1,1));
                end
            end

            [costToGo(4,4,1),FVal(4,4,1)] = max(q.*p.*J(4,3,1) + (1-q).*p.*J(3,4,1)+↙
(1-p).*J(4,4,2));
            [costToGo(4,4,2),FVal(4,4,2)] = max(q.*p.*J(4,3,1) + (1-q.*p).*J(3,4,1));

           % max(max(max(J-costToGo)))/max(max(max(costToGo)))
            if (max(max(max(J-costToGo)))/max(max(max(costToGo))) < err)
                % update cost
                J = costToGo;
                break;
```

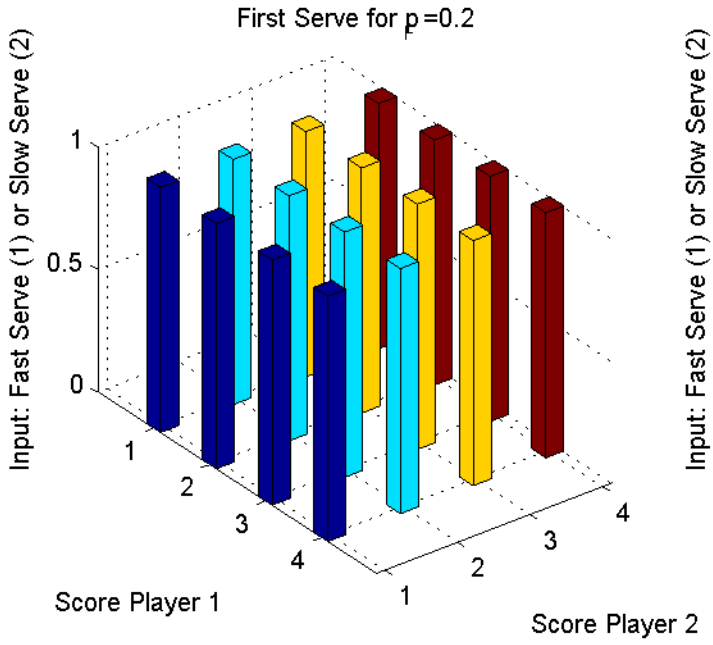
```
            else
                J = costToGo;
            end
        end


        % Probability of player 1 wining the game
        JValsave(m)=J(1, 1, 1);
end;



%% POLICY ITERATION
for m=1:mend
        %% PARAMETER
        % Landing probability
        if test == 2
            p(1) = (m-1)*incr; % Fast serve, check for 0...0.9
        end


        %% INITIALIZE PROBLEM
        % Our state space is S={0,1,2,3}x{0,1,2,3}x{1,2},
        % i.e. x_k = [score player 1, score player 2, serve]
        % ATTENTION: indices are shifted in code


        % Initialize the optimal control policy: 1 represents Fast serve, 2
        % represents Slow serve - in vector form
        FPol = 2.*ones(32,1);

        %% VALUE ITERATION

        iter = 0;
        while (1)
            iter = iter+1;
            disp(['Policy Iteration Number ',num2str(iter)]);
            if (mod(iter,1e3)==0)
                disp(['Policy Iteration Number ',num2str(iter)]);
            end

            % Update the value
            % Initialize matrices
            APol = zeros(32,32);
            bPol = zeros(32,1);
            for i = 1:3
                    % [costToGo(4,i,1),FVal(4,i,1)] = max(q.*p + (1-q).*p.*J(4,i+1,1)↙
+(1-p).*J(4,i,2));
                    f = FPol(12+i);
                    APol(12+i,12+i)=-1;
                    bPol(12+i)= -q(f)*p(f);
                    APol(12+i,13+i)= (1-q(f))*p(f);
                    APol(12+i,28+i)= (1-p(f));
                    % [costToGo(4,i,2),FPol(4,i,2)] = max(q.*p + (1-q.*p).*J(4,i+1,↙
1));
```

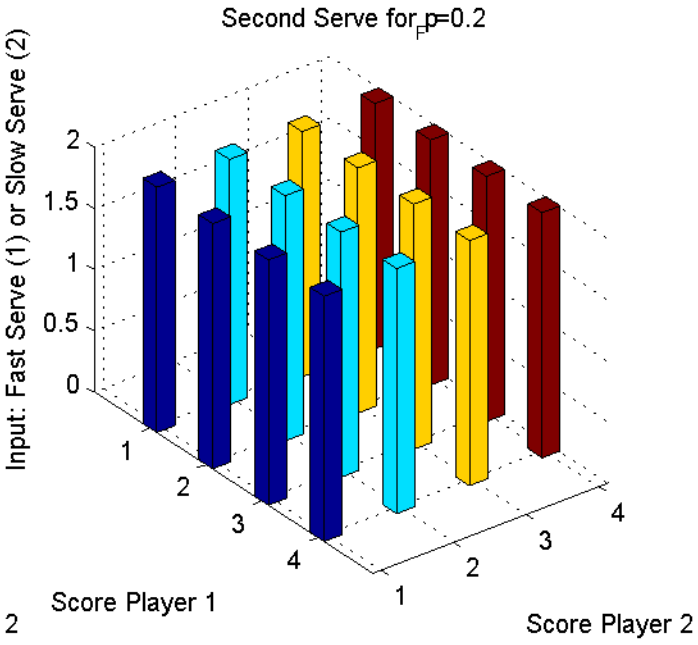
```
                    f = FPol(28+i);
                    APol(28+i,28+i)=-1;
                    bPol(28+i)= -q(f)*p(f);
                    APol(28+i,13+i)= 1-p(f)*q(f);
                    % [costToGo(i,4,1),FPol(i,4,1)] = max(q.*p.*J(i+1,4,1) + (1-p).*J↙
(i,4,2));
                    f = FPol(4*i);
                    APol(4*i,4*i)=-1;
                    APol(4*i,4*i+4)= q(f)*p(f);
                    APol(4*i,20+4*(i-1))= 1-p(f);
                    % [costToGo(i,4,2),FPol(i,4,2)] = max(q.*p.*J(i+1,4,1));
                    f = FPol(4*i+16);
                    APol(4*i+16,4*i+16)=-1;
                    APol(4*i+16,4*i+4)= q(f)*p(f);
                for j = 1:3
                    % Maximize over input 1 and 2
                    % [costToGo(i,j,1),FPol(i,j,1)] = max(q.*p.*J(i+1,j,1) + (1-q).↙
*p.*J(i,j+1,1)+(1-p).*J(i,j,2));
                    f = FPol(4*(i-1)+j);
                    APol(4*(i-1)+j,4*(i-1)+j)=-1;
                    APol(4*(i-1)+j,4*i+j)= q(f)*p(f);
                    APol(4*(i-1)+j,4*(i-1)+j+1)= (1-q(f))*p(f);
                    APol(4*(i-1)+j,4*(i-1)+j+16)= (1-p(f));
                    % [costToGo(i,j,2),FPol(i,j,2)] = max(q.*p.*J(i+1,j,1) + (1-q.↙
*p).*J(i,j+1,1));
                    f = FPol(4*(i-1)+j+16);
                    APol(4*(i-1)+j+16,4*(i-1)+j+16)=-1;
                    APol(4*(i-1)+j+16,4*i+j)= q(f)*p(f);
                    APol(4*(i-1)+j+16,4*(i-1)+j+1)= 1-q(f)*p(f);
                end
            end

            % [costToGo(4,4,1),FPol(4,4,1)] = max(q.*p.*J(4,3,1) + (1-q).*p.*J(3,4,1)↙
+(1-p).*J(4,4,2));
            f = FPol(16);
            APol(16,16)=-1;
            APol(16,15)= q(f)*p(f);
            APol(16,12)= (1-q(f))*p(f);
            APol(16,32)= (1-p(f));
            % [costToGo(4,4,2),FPol(4,4,2)] = max(q.*p.*J(4,3,1) + (1-q.*p).*J↙
(3,4,1));
            f = FPol(32);
            APol(32,32)=-1;
            APol(32,15)= q(f)*p(f);
            APol(32,12)= 1-q(f)*p(f);

            % Calculate cost
            JPol = zeros(32,1);
            JPol = APol\bPol;

            % Can now update the policy.
            FPolNew = FPol;
            JPolNew = JPol;

            JPol_re = reshape(JPol,4,4,2);
```

```
            JPol_re(:,:,1)= JPol_re(:,:,1)';
            JPol_re(:,:,2)= JPol_re(:,:,2)';

            % Initialize cost-to-go
            costToGo = zeros(4, 4, 2);
            FPolNew_re = zeros(4, 4, 2);

            % Update the optimal policy
            for i = 1:3
                    [costToGo(4,i,1),FPolNew_re(4,i,1)] = max(q.*p + (1-q).*p.↙
*JPol_re(4,i+1,1)+(1-p).*JPol_re(4,i,2));
                    [costToGo(4,i,2),FPolNew_re(4,i,2)] = max(q.*p + (1-q.*p).↙
*JPol_re(4,i+1,1));
                    [costToGo(i,4,1),FPolNew_re(i,4,1)] = max(q.*p.*JPol_re(i+1,4,1)↙
+ (1-p).*JPol_re(i,4,2));
                    [costToGo(i,4,2),FPolNew_re(i,4,2)] = max(q.*p.*JPol_re(i+1,↙
4,1));
                for j = 1:3
                    % Maximize over input 1 and 2
                    [costToGo(i,j,1),FPolNew_re(i,j,1)] = max(q.*p.*JPol_re(i+1,j,1)↙
+ (1-q).*p.*JPol_re(i,j+1,1)+(1-p).*JPol_re(i,j,2));
                    [costToGo(i,j,2),FPolNew_re(i,j,2)] = max(q.*p.*JPol_re(i+1,j,1)↙
+ (1-q.*p).*JPol_re(i,j+1,1));
                end
            end

            [costToGo(4,4,1),FPolNew_re(4,4,1)] = max(q.*p.*JPol_re(4,3,1) + (1-q).↙
*p.*JPol_re(3,4,1)+(1-p).*JPol_re(4,4,2));
            [costToGo(4,4,2),FPolNew_re(4,4,2)] = max(q.*p.*JPol_re(4,3,1) + (1-q.↙
*p).*JPol_re(3,4,1));

            JPolNew(1:16) = reshape(costToGo(:,:,1)', 16,1);
            JPolNew(17:32) = reshape(costToGo(:,:,2)', 16,1);
            FPolNew(1:16) = reshape(FPolNew_re(:,:,1)', 16,1);
            FPolNew(17:32) = reshape(FPolNew_re(:,:,2)', 16,1);

            % Quit if the policy is the same, or if the costs are essentially the
            % same.
            if ((norm(JPol - JPolNew)/norm(JPol) < 1e-10) )
                break;
            else
                FPol = FPolNew;
            end
        end


        % Probability of player 1 wining the game
        JPolsave(m)=JPolNew(1,1);


end;
```

```
if test ==2
   plot(0:incr:(0.95-incr), JValsave,'.');
   hold on
   plot(0:incr:(0.95-incr), JPolsave,'r');
   grid on
   title('Probability of the server winning a game')
   xlabel('p_F')
   ylabel('Probability of winning')
   legend('Value Iteration', 'Policy Iteration')
else
    disp(['Probability of player 1 winning ',num2str(JValsave(1))]);

    subplot(1,2,1)
    bar3(FVal(:,:,1),0.25,'detached')
    title(['First Serve for p_F =', num2str(p(1))])
    ylabel('Score Player 1')
    xlabel('Score Player 2')
    zlabel('Input: Fast Serve (1) or Slow Serve (2)')


    subplot(1,2,2)
    bar3(FVal(:,:,2),0.25,'detached')
    title(['Second Serve for p_F =', num2str(p(1))])
    ylabel('Score Player 1')
    xlabel('Score Player 2')
    zlabel('Input: Fast Serve (1) or Slow Serve (2)')
end
```


Probability of the server winning a game
Value Iteration
Policy Iteration
Probability of winning
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
p_F


First Serve for p =0.2
Input: Fast Serve (1) or Slow Serve (2)
1
0.5
0
1
2
3
4
Score Player 1
1
2
3
4
Score Player 2


Second Serve for p=0.2
Input: Fast Serve (1) or Slow Serve (2)
2
1.5
1
0.5
0
1
2
3
4
Score Player 1
1
2
3
4
Score Player 2

11.11.09

Problem 7.3

- states: $i \in S = \{1, 2\}$, 1: sells well, 2: doesn't sell well
- decision / control: $\mu(1) \in U(1) = \{A, \bar{A}\}$
  A: advertise
  $\bar{A}$: don't advertise

  $\mu(2) \in U(2) = \{R, \bar{R}\}$
  R: do research
  $\bar{R}$: don't do research
- rewards and transition probabilities

$i=1, \mu(1)=A$:	$g(1,A) = 4$	$p_{11}(A) = 0.8$	$p_{12}(A) = 0.2$
$i=1, \mu(1)=\bar{A}$:	$g(1,\bar{A}) = 6$	$p_{11}(\bar{A}) = 0.5$	$p_{12}(\bar{A}) = 0.5$
$i=2, \mu(2)=R$:	$g(2,R) = -5$	$p_{21}(R) = 0.7$	$p_{22}(R) = 0.3$
$i=2, \mu(2)=\bar{R}$:	$g(2,\bar{R}) = -3$	$p_{21}(\bar{R}) = 0.4$	$p_{22}(\bar{R}) = 0.6$

a) • Approach: We consider all possible stationary policies μ^i and compute the cost J^i. Then, by Proposition 7.3.1, the policy with the largest reward is an optimal policy since it attains the maximum in Bellman's equation.

Define policy $\mu^i := (\mu(1), \mu(2))$.
→ 4 different stationary policies: (A,R), $(A,\bar{R})$, $(\bar{A},R)$, $(\bar{A},\bar{R})$

$\underline{\mu^1 = (A,R)}$:

$$J(1) = g(1,A) + \alpha\left(p_{11}(A)\cdot J(1) + p_{12}(A)\cdot J(2)\right)$$
$$J(1) = 4 + \alpha\left(0.8\cdot J(1) + 0.2\, J(2)\right)$$

$$J(2) = -5 + \alpha\left(0.7\, J(1) + 0.3\, J(2)\right)$$

Combine into single vector equation:

$$\underbrace{\begin{bmatrix} J(1) \\ J(2) \end{bmatrix}}_{=: J^1} = \begin{bmatrix} 4 \\ -5 \end{bmatrix} + \alpha \begin{bmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{bmatrix} \begin{bmatrix} J(1) \\ J(2) \end{bmatrix}$$

$$\Leftrightarrow \vec{J}^1 = \left(I - \alpha \begin{bmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{bmatrix}\right)^{-1} \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

Similarly, obtain the costs for other policies.

$\underline{\mu^2 = (A, R)}$:

$$\vec{J}_4^2 = \left(I - \alpha \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}\right)^{-1} \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

$\underline{\mu^3 = (\bar{A}, R)}$:

$$\vec{J}^3 = \left(I - \alpha \begin{bmatrix} 0.5 & 0.5 \\ 0.7 & 0.3 \end{bmatrix}\right)^{-1} \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

$\underline{\mu^4 = (\bar{A}, \bar{R})}$:

$$\vec{J}^4 = \left(I - \alpha \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix}\right)^{-1} \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

Reformulate inverse $(I - \alpha M)^{-1}$ such that we can then take limits. M has a special structure:

$$M = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

$$\rightarrow (I - \alpha M) = \begin{bmatrix} 1-\alpha+\alpha p & -\alpha p \\ -\alpha q & 1-\alpha+\alpha q \end{bmatrix}$$

$$(I - \alpha M)^{-1} = \frac{1}{\underset{-\alpha^2 pq}{(1-\alpha+\alpha p)(1-\alpha+\alpha q)}} \begin{bmatrix} 1-\alpha+\alpha q & \alpha p \\ \alpha q & 1-\alpha+\alpha p \end{bmatrix}$$

Denominator: $1 - \alpha + \alpha q - \alpha + \alpha^2 - \alpha^2 q + \alpha p - \alpha^2 p + \alpha^2 pq - \alpha^2 pq$

$$= 1 - 2\alpha + \alpha^2 + \alpha q + \alpha p - \alpha^2 q - \alpha^2 p$$
$$= (1-\alpha)^2 + (1-\alpha)\alpha(q+p)$$
$$= (1-\alpha)(1 - \alpha + \alpha(q+p))$$

$$\Rightarrow \lim_{\alpha \to 0} (I - \alpha M)^{-1} = I \quad , \quad \lim_{\alpha \to 1} (I - \alpha M)^{-1} = \frac{1}{(1-\alpha)(p+q)} \begin{bmatrix} q & p \\ q & p \end{bmatrix}$$

Case 1: α sufficiently small: $\alpha \to 0$

$$\lim_{\alpha\to 0} \bar{J}^1 = \begin{bmatrix} 4 \\ -5 \end{bmatrix}, \quad \lim_{\alpha\to 0} \bar{J}^2 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

$$\lim_{\alpha\to 0} \bar{J}^3 = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \quad \lim_{\alpha\to 0} \bar{J}^4 = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

"element-wise"

$$\Rightarrow \lim_{\alpha\to 0} \bar{J}^4 \geq \lim_{\alpha\to 0} \bar{J}^j, \quad j = 1, \dots, 3$$

$\Rightarrow$ "shortsighted" policy $\mu^4 = (\bar{A}, \bar{R})$ is optimal

Case 2: α sufficiently close to 1: $\alpha \to 1$

$$\lim_{\alpha\to 1} \bar{J}^1 = \frac{1}{(1-\alpha)0.9} \begin{bmatrix} 0.7 & 0.2 \\ 0.7 & 0.2 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \frac{1.8}{0.9} \frac{1}{1-\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= 2 \cdot \frac{1}{1-\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lim_{\alpha\to 1} \bar{J}^2 = \frac{1.0}{0.6} \frac{1}{1-\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{5}{3} \frac{1}{1-\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lim_{\alpha\to 1} \bar{J}^3 = \frac{4.2 - 2.5}{1.2} \frac{1}{1-\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{17}{12} \frac{1}{1-\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lim_{\alpha\to 1} \bar{J}^4 = \frac{0.9}{0.9} \frac{1}{1-\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \frac{1}{1-\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \lim_{\alpha\to 1} \bar{J}^1 \geq \lim_{\alpha\to 1} \bar{J}^j, \quad j = 2, 3, 4$$

$\Rightarrow$ "farsighted" policy $\mu^1 = (\bar{A}, R)$ is optimal

b) Policy Iteration with $\alpha = 0.9$

- Start iteration with stationary policy $\mu^0 = (\mu^0(1), \mu^0(2)) = (\bar{A}, \bar{R})$

- Step 1: Policy Evaluation Step

$$J(1) = g(1, \mu^0(1)) + \alpha \left(p_{11}(\mu^0(1)) \cdot J(1) + p_{12}(\mu^0(1)) \cdot J(2) \right)$$
$$J(2) = g(2, \mu^0(2)) + \alpha \left(p_{21}(\mu^0(2)) \cdot J(1) + p_{22}(\mu^0(2)) \, J(2) \right)$$

$$J_{\mu_0} = \begin{bmatrix} J(1) \\ J(2) \end{bmatrix} = \begin{bmatrix} g(1, \mu^0(1)) \\ g(2, \mu^0(2)) \end{bmatrix} + \alpha \begin{bmatrix} p_{11}(\mu^0(1)) & p_{12}(\mu^0(1)) \\ p_{21}(\mu^0(2)) & p_{22}(\mu^0(2)) \end{bmatrix} J_{\mu_0}$$

$$\Rightarrow J_{\mu_0} = \left(I - \alpha \begin{bmatrix} p_{11}(\mu_0(1)) & p_{12}(\mu^0(1)) \\ p_{21}(\mu_0(2)) & p_{22}(\mu^0(2)) \end{bmatrix} \right)^{-1} \begin{bmatrix} g(1, \mu^0(1)) \\ g(2, \mu^0(2)) \end{bmatrix}$$

$$\underline{J_{\mu_0}} = \left(I - \alpha \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 6 \\ -3 \end{bmatrix} \approx \underline{\begin{bmatrix} 15.49 \\ 5.60 \end{bmatrix}}$$

- Step 2: Policy Improvement Step

$$\mu^{k+1}(i) = \arg\max_{u \in U(i)} \left(g(i,u) + \alpha \sum_{j=1}^{2} p_{ij}(u) \, J_{\mu^k}(j) \right) \qquad i = 1, 2$$

$$\mu^1(1) = \arg\max \left\{ 4 + 0.9 \left(0.8 \cdot J_{\mu_0}(1) + 0.2 \, J_{\mu_0}(2) \right), \right.$$
$$\left. 6 + 0.9 \left(0.5 \, J_{\mu_0}(1) + 0.5 \, J_{\mu_0}(2) \right) \right\}$$

$$= \arg\max \{ 16.2, 15.5 \} = A$$

$$\mu^1(2) = \arg\max \left\{ -5 + 0.9 \left(0.7 \cdot J_{\mu_0}(1) + 0.3 \cdot J_{\mu_0}(2) \right), \right.$$
$$\left. -3 + 0.9 \left(0.4 \, J_{\mu_0}(1) + 0.6 \, J_{\mu_0}(2) \right) \right\}$$

$$= \arg\max \{ 6.27, 5.6 \} = R$$

$\rightarrow \underline{\mu^1 = (A, R)}$

- Step 1: Policy Evaluation Step #2

$$\underline{J_{\mu^1}} = \left(I - 0.9 \begin{bmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{bmatrix}\right)^{-1} \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \underline{\begin{bmatrix} 22.20 \\ 12.31 \end{bmatrix}}$$

- Step 2: Policy improvement Step #2

$$\mu^2(1) = \arg\max \{ 4 + 0.9 (0.8\, J_{\mu^1}(1) + 0.2\, J_{\mu^1}(2)),$$
$$6 + 0.9 (0.5\, J_{\mu^1}(1) + 0.5\, J_{\mu^1}(2)) \}$$
$$= \arg\max \{ 22.2, 21.53 \} = A$$

$$\mu^2(2) = \arg\max \{ -5 + 0.9 (0.7\, J_{\mu^1}(1) + 0.3\, J_{\mu^1}(2)),$$
$$-3 + 0.9 (0.4\, J_{\mu^1}(1) + 0.6\, J_{\mu^1}(2)) \}$$
$$= \arg\max \{ 12.31, 11.64 \} = R$$

$\rightarrow \underline{\mu^2 = (A, R)}$

- Next policy evaluation step would yild the same result, i.e.

$$J_{\mu^2}(i) = J_{\mu^1}(i) \quad \forall\, i = 1, 2$$

Therefore, we can terminate. The optimal policy is (A, R).

c) Implementation in Matlab: see printout

Result for $\alpha = 0.99$:

$J^*(1) = 202.22$, $J^*(2) = 192.23$

$\mu^* = (A, R)$.

```
% script_P73c.m
%
% Matlab Script with implementation of a solution to Problem 7.3c from the
% class textbook.
%
% Dynamic Programming and Optimal Control
% Fall 2009
% Problem Set, Chapter 7
% Problem 7.3 c
%
% --
% ETH Zurich
% Institute for Dynamic Systems and Control
% Sebastian Trimpe
% strimpe@ethz.ch
%
% --
% Revision history
% [11.11.09, ST]    first version
%
%


% clear workspace and command window
clear;
%clc;


%% Program control
% Flag defining whether we run the value iteration with or without error
% bounds.
flag_errbnds = true;
% False: No error bounds; fixed number of iterations MAX_ITER.
% True: Use error bounds.  Terminate when result within tolerance ERR_TOL.

MAX_ITER = 1000;
ERR_TOL = 1e-6;


%% Setup problem data
% Stage costs g.  g is a 2x2 matrix where the first index correponds to the
% state i=1,2 and the second index correponds to the admissible control
% input u=1 (=advertising/do research), or 2 (=don't advertise/don't do
% research).
g = [4 6; -5 -3];

% Transition probabilities.  P is a 2x2x2 matrix, where the first index
% corresponds to the origin state, the second index corresponds to the
% destination state and the third input corresponds to the applied control
% input where (1,2) maps to (advertise/do research, don't advertise/don't
% do research).  For example, the probability of transition from node 1 to
% node 2 given that we do not advertice is P(1,2,2).
P = zeros(2,2,2);

% Advertise/do research (u=1):
```

```
P(:,:,1) = [0.8 0.2; 0.7 0.3];

% Don't advertise/don't do research (u=2):
P(:,:,2) = [0.5 0.5; 0.4 0.6];

% Discount factor.
%alpha = 0.9;
alpha = 0.99;


%% Value Iteration
% Initialize variables that we update during value iteration.
% Cost (here it really is the reward):
costJ = [0,0];
costJnew = [0,0];

% Policy
policy = [0,0];

% Control variable for breaking the loop
doBreak = false;


% Loop over value iterations k.
for k=1:MAX_ITER

    for i=1:2   % loop over two states
        % One value iteration step for each state.
        [costJnew(i),policy(i)] = max( g(i,:) + alpha*costJ*squeeze(P(i,:,:)) );
    end;

    % Save results for plotting later.
    res_J(k,:) = costJnew;

    % Construct string to be displayed later:
    dispstr = ['k=',num2str(k,'%5d'), ...
        '   J(1)=',num2str(costJnew(1),'%6.4f'), ...
        '   J(2)=',num2str(costJnew(2),'%6.4f'), ...
        '   mu(1)=',num2str(policy(1),'%3d'), ...
        '   mu(2)=',num2str(policy(2),'%3d')];

    % Use error bounds if flag has been set.
    if flag_errbnds

        % Compute error bounds.
        lower = costJnew + alpha/(1-alpha)*min(costJnew-costJ); % lower bound on J*
        upper = costJnew + alpha/(1-alpha)*max(costJnew-costJ); % upper bound on J*

        % display string:
        dispstr = [dispstr,'   lower=[',num2str(lower,'%9.4f'), ...
            ']   upper=[',num2str(upper,'%9.4f'),']'];

        % If desired tolerance reached: can terminate the algorithm.
        if all((upper-lower)<=ERR_TOL)
            % Difference between upper and lower bound less than specified
```

```
            % tolerance for all states: set cost to average of upper and
            % lower bound and terminate.
            % Note that using the bounds, we can get a better result than
            % just using J_{k+1}.
            costJnew = mean([lower; upper],1);

            dispstr = [dispstr, ...
                sprintf('\nResult within desired tolerance. Terminate.\n')];
            doBreak = true;
        end;

        % Save results for plotting later.
        res_lower(k,:) = lower;
        res_upper(k,:) = upper;
    end;

    % Update the cost.  One could also implement an immidiate update of the
    % cost as Gauss-Seidel type update, which would yield faster
    % convergence.
    costJ = costJnew;

    % Display:
    disp(dispstr);

    if doBreak
        break;
    end;
end;


% display the optained costs
disp('Result:');
disp(['  J*(1) = ',num2str(costJ(1),'%9.6f')]);
disp(['  J*(2) = ',num2str(costJ(2),'%9.6f')]);
disp(['  mu*(1) = ',num2str(policy(1))]);
disp(['  mu*(2) = ',num2str(policy(2))]);



%% Plots
% Plot costs and bounds over iterations.
kk = 1:size(res_J,1);

figure;
subplot(2,1,1);
if(flag_errbnds)
    plot(kk,[res_J(:,1),res_lower(:,1), res_upper(:,1)], ...
        [kk(1),kk(end)],[costJ(1), costJ(1)],'k');
else
    plot(kk,[res_J(:,1)],[kk(1),kk(end)],[costJ(1), costJ(1)],'k');
end;
grid;
%legend('lower bound','upper bound','J_k','J*');
%xlabel('iteration');
%
```

```
subplot(2,1,2);
if(flag_errbnds)
    plot(kk,[res_J(:,2),res_lower(:,2), res_upper(:,2)], ...
        [kk(1),kk(end)],[costJ(2),costJ(2)],'k');
else
    plot(kk,[res_J(:,2)],[kk(1),kk(end)],[costJ(2), costJ(2)],'k');
end;
grid;
if(flag_errbnds)
    legend('J_k','lower bound','upper bound','J*');
else
    legend('J_k','J*');
end;
xlabel('iteration');
```

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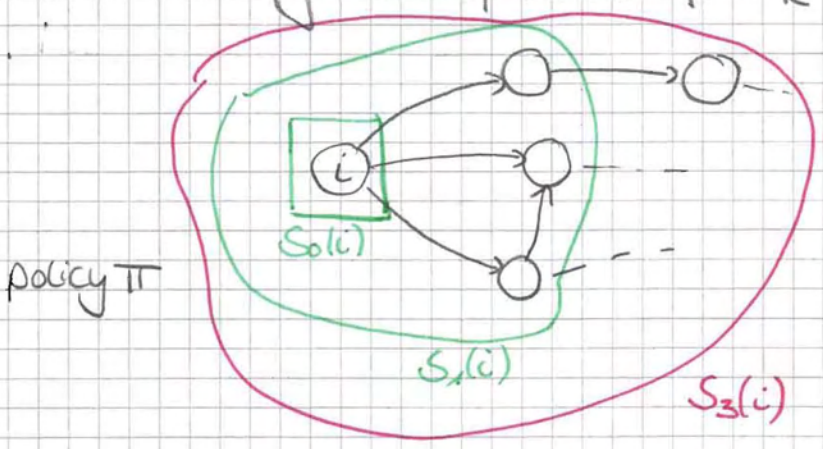
Exercise 7.12

Consider the given definition of $S_k(i)$ for any given π and i.



- Note:
  - $S_0(i) = \{i\}$
  - $S_k(i) \subseteq S_{k+1}(i)$ subset

- Assumption 7.2.1:
  For any π and i, $t \in S_m(i)$. (*)

- To be shown:
  For any π and i, $t \in S_n(i)$ where n is the number of nodes (not including the termination state) (**)

Proof 1

Assume for some $k<n$, $t \notin S_k(i)$ [for any π and i].
Because of (*), we know that $\exists\ j \in S_k(i)$ for which we end up at t after $(n-k)$ steps. That is,

$$S_k(i) \subset S_{k+1}(i)$$ (strict or proper subset)

as long as $t \notin S_k(i)$.

However, that means after at most n steps, we have "explored" the whole state space and already reached t with positive probability.

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Proof 2 - by contradiction

- Assume (**) is not true.

  ⇕ equivalent to

  For some k, $k < n$, $S_k(i) = S_{k+1}(i) \not\ni t$ [for some π and i]

  ⇕

  $P_{jL}(\mu) = 0$, $\forall j \in S_k(i)$ and $L \notin S_k(i)$

  This is, the termination state is not reachable from any $j \in S_k(i)$ what is a contradiction to (*)

Problem 1.23

[the monotonicity property of DP is used in proof of policy iteration on page 415.]

- Given $J_{N-1}(x) \leq J_N(x) \quad \forall x \in S$.
- To be proven: $J_k(x) \leq J_{k+1}(x) \quad \forall x \in S \quad \forall k$

Proof by Induction

- Start: $J_{N-1}(x) \leq J_N(x) \quad \forall x \in S$ given
- Hypothesis: Assume $J_k(x) \leq J_{k+1}(x) \quad \forall x \in S$ for some k
- Induction step:
  - Let $C_k(x,u) := \underset{w}{E}\left(g(x,u,w) + J_{k+1}\left(f(x,u,w)\right)\right)$.

    Then $C_k(x,u) = \underset{w}{E}\left(g(x,u,w)\right) + \underset{w}{E}\left(J_{k+1}\left(f(x,u,w)\right)\right)$

    $\geq \underset{w}{E}\left(g(x,u,w)\right) + \underset{w}{E}\left(J_k\left(f(x,u,w)\right)\right)$ (by induction hypothesis)

    $= C_{k-1}(x,u)$

    $\Rightarrow \quad C_{k-1}(x,u) \leq C_k(x,u) \quad \forall x,u$

  - Therefore,

    $J_{k-1}(x) = \min_{u \in U(x)} \underset{w}{E}\left(g(x,u,w) + J_k\left(f(x,u,w)\right)\right)$

    $= \min_{u \in U(x)} C_{k-1}(x,u)$

    $\leq \min_{u \in U(x)} C_k(x,u)$

    ($C_{k-1}(x,u) \leq C_k(x,u) \ \forall u$, therefore taking the minimum over some range $u \in U(x)$ preserves the order: $\min_{u \in U(x)} C_{k-1}(x,u) \leq \min_{u \in U(x)} C_k(x,u)$)

    $= J_k(x)$

$\Rightarrow J_{k-1}(x) \leq J_k(x) \quad \forall x$ q.e.d.

The proof for the reversed order, i.e. given $f_{N-1}(x) \geq f_N(x) \quad \forall x$, prove $f_k(x) \geq f_{k+1}(x) \quad \forall x, k$ goes the same way; just reverse $\geq / \leq$ everywhere.