ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute for
Dynamic Systems and Control
IDSC
Institut für Dynamische Systeme
und Regelungstechnik

Dynamic Programming and Optimal Control

Fall 2009

Problem Set:
The Dynamic Programming Algorithm

Notes:

- Problems marked with BERTSEKAS are taken from the book *Dynamic Programming and Optimal Control by Dimitri P. Bertsekas, Vol. I, 3rd edition, 2005, 558 pages, hardcover.*
- The solutions were derived by the teaching assistants in the previous class. Please report any error that you may find to `strimpe@ethz.ch` or `aschoellig@ethz.ch`.

Problem Set

Problem 1 (BERTSEKAS, p. 51, exercise 1.1 a, c)

Consider the system

$$x_{k+1} = x_k + u_k + w_k, \qquad k = 0, 1, 2, 3,$$

with initial state $x_0 = 5$, and the cost function

$$\sum_{k=0}^{3} (x_k^2 + u_k^2).$$

Apply the DP algorithm for the following two cases:

a) The control constraint set $U_k(x_k)$ is $\{u \mid 0 \leq x_k + u \leq 5, u : \text{integer}\}$ for all x_k and k, and the disturbance w_k is equal to zero for all k.

b) The control constraint is as in part (a) and the disturbance w_k takes the values -1 and 1 with equal probability $\frac{1}{2}$ for all x_k and u_k, except if $x_k + u_k$ is equal to 0 or 5, in which case $w_k = 0$ with probability 1.

Problem 2 (BERTSEKAS, p. 52, exercise 1.3)

Suppose we have a machine that is either running or is broken down. If it runs throughout one week, it makes a gross profit of \$100. If it fails during the week, gross profit is zero. If it is running at the start of the week and we perform preventive maintenance, the probability that it will fail during the week is 0.4. If we do not perform such maintenance, the probability of failure is 0.7. However, maintenance will cost \$20. When the machine is broken down at the start of the week, it may either be repaired at cost of \$40, in which case it will fail during the week with a probability of 0.4, or it may be replaced at a cost of \$150 by a new machine that is guaranteed to run through its first week of operation. Find the optimal repair, replacement, and maintenance policy that maximizes total profit over four weeks, assuming a new machine at the start of the first week.

Problem 3 (Discounted Cost per Stage, BERTSEKAS, p. 53, exercise 1.6)

In the framework of the basic problem, consider the case where the cost is of the form

$$\mathop{\mathrm{E}}_{\substack{w_k \\ k=0,1,\ldots,N-1}} \left\{ \alpha^N g_N(x_N) + \sum_{k=0}^{N-1} \alpha^k g_k(x_k, u_k, w_k) \right\},$$

where α is a discount factor with $0 < \alpha < 1$. Show that an alternative form of the DP algorithm is given by

$$V_N(x_N) = g_N(x_N),$$

$$V_k(x_k) = \min_{u_k \in U_k(x_k)} \mathop{\mathrm{E}}_{w_k} \left\{ g_k(x_k, u_k, w_k) + \alpha V_{k+1}\Big(f_k(x_k, u_k, w_k)\Big) \right\}.$$

Problem 4 (Exponential Cost Function, BERTSEKAS, p. 53, exercise 1.7)

In the framework of the basic problem, consider the case where the cost is of the form

$$\underset{\substack{w_k\\k=0,1,\ldots,N-1}}{\mathrm{E}}\left\{\exp\left(g_N(x_N)+\sum_{k=0}^{N-1}g_k(x_k,u_k,w_k)\right)\right\}.$$

.

a) Show that the optimal cost and optimal policy can be obtained from the DP-like algorithm

$$J_N(x_N)=\exp\big(g_N(x_N)\big),$$

$$J_k(x_k)=\min_{u_k\in U_k(x_k)}\underset{w_k}{\mathrm{E}}\Big\{J_{k+1}\big(f_k(x_k,u_k,w_k)\big)\exp\big(g_k(x_k,u_k,w_k)\big)\Big\}.$$

b) Define the function $V_k(x_k)=\ln J_k(x_k)$. Assume also that g_k is a function of x_k and u_k only (and not of w_k). Show that the above algorithm can be rewritten as

$$V_N(x_N)=g_N(x_N),$$

$$V_k(x_k)=\min_{u_k\in U_k(x_k)}\left\{g_k(x_k,u_k)+\ln\underset{w_k}{\mathrm{E}}\left\{\exp\Big(V_{k+1}\Big(f_k(x_k,u_k,w_k)\Big)\Big)\right\}\right\}.$$

Note: the exponential cost function is an example of a *risk-sensitive cost function* that can be used to encode a preference for policies with a small variance of the cost $g_N(x_N)+\sum_{k=0}^{N-1}g_k(x_k,u_k,w_k)$. The associated problems have a lot of interesting properties, which are discussed in several sources, e.g. Whittle [Whi90], Fernandez-Gaucherand and Markus [FeM94], James, Baras, and Elliott [JBE94]. Basar and Bernhard [BaB95].

Problem 5 (Terminating Process, BERTSEKAS, p. 54, exercise 1.8)

In the framework of the basic problem, consider the case where the system evolution terminates at time i when a given value $\overline{w}_i$ of the disturbance at time i occurs, or when a termination decision $\overline{u}_i$ is made by the controller. If termination occurs at time i, the resulting cost is

$$T+\sum_{k=0}^{i}g_k(x_k,u_k,w_k),$$

where T is a termination cost. If the process has not terminated up to the final time N, the resulting cost is $g_N(x_N)+\sum_{k=0}^{N-1}g_k(x_k,u_k,w_k)$. Reformulate the problem into the framework of the basic problem. *Hint*: Augment the state space with a special termination state.

Problem 6 (Inscribed Polygon of Maximal Perimeter, BERTSEKAS, p. 59, exercise 1.22)

Consider the problem of inscribing an N-side polygon in a given circle, so that the polygon has maximal perimeter.

a) Formulate the problem as a DP problem involving sequential placement of N points in the circle.

b) Use DP to show that the optimal polygon is regular (all sides are equal).

Sample Solutions

Problem 1 (Solution)

System:

$$x_{k+1} = x_k + u_k + w_k, \qquad k = 0, 1, 2, 3$$

Cost to minimize:

$$\sum_{k=0}^{3} (x_k^2 + u_k^2).$$

a)
- $w_k = 0$ (no disturbance)
- control constraint set $U_k(x_k) := \{u \mid 0 \leq x_k + u \leq 5, u : \text{integer}\}$

$$\begin{aligned} & 0 \leq x_k + u_k \leq 5 \qquad \forall k \\ \Leftrightarrow \quad & 0 \leq x_{k+1} \leq 5 \qquad \forall k \end{aligned}$$

with $x_0 = 5 \quad \Rightarrow \quad 0 \leq x_k \leq 5 \qquad \forall k$

$\Rightarrow$ states only take the values $0, \ldots, 5$

- $N = 4$

Apply the Dynamic Programming Algorithm (DPA)

- $\underline{k = N}$

$$J_N(x_N) = 0 = J_4(x_4)$$

- $\underline{k = 3}$

$$\begin{aligned} J_3(x_3) &= \min_{-x_3 \leq u_3 \leq 5 - x_3} \left(x_3^2 + u_3^2 + J_4(x_3 + u_3) \right) \\ &= \min_{-x_3 \leq u_3 \leq 5 - x_3} \left(x_3^2 + u_3^2 + 0 \right) \end{aligned}$$

$\Rightarrow$ optimal control: $\underline{\underline{u_3 = \mu_3(x_3) = 0}}$

$\Rightarrow$ $\underline{\underline{J_3(x_3) = x_3^2}}$

- $\underline{k = 2}$

$$\begin{aligned} J_2(x_2) &= \min_{-x_2 \leq u_2 \leq 5 - x_2} \left(x_2^2 + u_2^2 + J_3(x_2 + u_2) \right) \\ &= \min_{-x_2 \leq u_2 \leq 5 - x_2} \left(2x_2^2 + 2x_2 u_2 + 2u_2^2 \right) \end{aligned}$$

Evaluate expression for all possible x_2, u_2:

	$u_2 = -5$	-4	-3	-2	-1	0	1	2	3	4	5
$x_2 = 0$	-	-	-	-	-	0	2	8	18	32	50
1	-	-	-	-	2	2	6	14	24	42	-
2	-	-	-	8	6	8	14	24	38	-	-
3	-	-	18	14	14	18	26	38	-	-	-
4	-	32	26	24	26	32	42	-	-	-	-
5	50	42	38	38	42	50	-	-	-	-	-

Therefore, the optimal cost and policy:

x_2	$J_2(x_2)$	$\mu_2(x_2)$
0	0	0
1	2	-1 or 0
2	6	-1
3	14	-2 or -1
4	24	-2
5	38	-3 or -2

- $\underline{k=1}$

$$J_1(x_1) = \min_{-x_1 \leq u_1 \leq 5-x_1} \left(x_1^2 + u_1^2 + J_2(x_1+u_1)\right)$$

	$u_1 = -5$	-4	-3	-2	-1	0	1	2	3	4	5
$x_1 = 0$	-	-	-	-	-	0	3	10	23	40	63
1	-	-	-	-	2	3	8	19	34	55	-
2	-	-	-	8	7	10	19	32	51	-	-
3	-	-	18	15	16	23	34	51	-	-	-
4	-	32	27	26	31	40	55	-	-	-	-
5	50	43	40	43	50	63	-	-	-	-	-

x_1	$J_1(x_1)$	$\mu_1(x_1)$
0	0	0
1	2	-1
2	7	-1
3	15	-2
4	26	-2
5	40	-3

- $\underline{k=0}$

$$J_0(x_0) = \min_{-x_0 \leq u_0 \leq 5-x_0} \left(x_0^2 + u_0^2 + J_1(x_0+u_0)\right)$$

given: $x_0 = 5$

$$J_0(5) = \min_{-5 \leq u_0 \leq 0} \left(25 + u_0^2 + J_1(5+u_0)\right)$$

x_0	-5	-4	-3	-2	-1	0
5	50	43	41	44	52	65

$$\rightarrow \quad \mu_0(x_0 = 5) = -3 \quad , \quad J_0(x_0 = 5) = 41$$

System evolution:

$x_0 = 5$	$\rightarrow$	$u_0 = -3$	$g_0 = 34$
$x_1 = 2$	$\rightarrow$	$u_1 = -1$	$g_1 = 5$
$x_2 = 1$	$\rightarrow$	$u_2 = -1$ or 0	$g_2 = 2$ or 1
$x_3 = 0$ or 1	$\rightarrow$	$u_3 = 0$	$g_3 = 0$ or 1

b)

$$\sum_{k=0}^{3} (x_k^2 + u_k^2)$$

$$x_{k+1} = x_k + u_k + w_k \quad , \quad u_k \in U_k(x_k) := \{u \mid 0 \leq x_k + u \leq 5, u : \text{integer}\}$$

- As noted in a) $x_k + u_k$ is always between 0 and 5.
- In case where $x_k + u_k$ equals 0 or 5, $w_k = 0$; otherwise w_k can take values $\{-1, 1\}$. Thus x_{k+1} also takes values $0 \ldots 5$ only.

Apply DPA

- $\underline{k = N}$

$$J_4(x_4) = 0$$

- $\underline{k = 3}$

$$\begin{aligned} J_3(x_3) &= \min_{-x_3 \le u_3 \le 5 - x_3} \operatorname*{E}_{w_3} \left(x_3^2 + u_3^2 + J_4(x_4) \right) \\ &= \min_{-x_3 \le u_3 \le 5 - x_3} \left(x_3^2 + u_3^2 \right) \\ &= x_3^2 \end{aligned}$$

$$\rightarrow \quad \underline{\underline{\mu_3(x_3) = 0}} \quad , \quad \underline{\underline{J_3(x_3) = x_3^2}}$$

- $\underline{k = 2}$

$$J_2(x_2) = \min_{-x_2 \le u_2 \le 5 - x_2} \operatorname*{E}_{w_2} \left(x_2^2 + u_2^2 + J_3(x_2 + u_2 + w_2) \right)$$

Case $u_2 + x_2 = 5$ or $u_2 + x_2 = 0$:
$$J_2(x_2) = \min \left(x_2^2 + u_2^2 + J_3(x_2 + u_2) \right)$$

Case $u_2 + x_2 \neq 5$ and $u_2 + x_2 \neq 0$:
$$J_2(x_2) = \min \left(x_2^2 + u_2^2 + \tfrac{1}{2} J_3(x_2 + u_2 + 1) + \tfrac{1}{2} J_3(x_2 + u_2 - 1) \right)$$

x_2	$J_2(x_2)$	$\mu_2(x_2)$
0	0	0
1	2	-1
2	7	-1
3	15	-2 or -1
4	25	-2
5	39	-2 or -3

- $\underline{k = 1}$

$$J_1(x_1) = \min_{-x_1 \le u_1 \le 5 - x_1} \operatorname*{E}_{w_1} \left(x_1^2 + u_1^2 + J_2(x_1 + u_1 + w_1) \right)$$

Case $u_2 + x_2 = 5$ or $u_2 + x_2 = 0$:
$$J_1(x_1) = \min \left(x_1^2 + u_1^2 + J_2(x_1 + u_1) \right)$$

Case $u_2 + x_2 \neq 5$ and $u_2 + x_2 \neq 0$:
$$J_1(x_1) = \min \left(x_1^2 + u_1^2 + \tfrac{1}{2} J_2(x_1 + u_1 + 1) + \tfrac{1}{2} J_2(x_1 + u_1 - 1) \right)$$

x_1	$J_1(x_1)$	$\mu_1(x_1)$
0	0	0
1	2	-1
2	8	-2
3	16.5	-2
4	28.5	-3 or -2
5	42.5	-3

- $\underline{k=0}$

$$J_0(x_0) = \min_{-x_0 \le u_0 \le 5-x_0} \mathrm{E}\big(x_0^2 + u_0^2 + J_1(x_0+u_0+w_0)\big)$$

$$\rightarrow \quad \underline{\underline{\mu_0(x_0=5) = -3}} \quad , \quad \underline{\underline{J_0(x_0=5) = 43.25}}$$

Problem 2 (Solution)

- States x: R : machine running, B : machine broken
- Control actions u:

$$\left.\begin{array}{ll} n: & \text{no maintenance} \\ m: & \text{maintenance} \end{array}\right\} \text{if } x = R$$

$$\left.\begin{array}{ll} r: & \text{repair} \\ l: & \text{replace} \end{array}\right\} \text{if } x = B$$

- Costs C:

$$\begin{array}{lll} n & \rightarrow & 0 \\ m & \rightarrow & 20 \\ r & \rightarrow & 40 \\ l & \rightarrow & 150 \end{array}$$

- Gain ($\triangleq$ negative cost): -100 if not broken

<u>Week 3</u>

$$\begin{array}{llll} x_3 = R & u_3 = n & C = 0.7(0) + 0.3(-100) & = -30 \\ & u_3 = m & C = 20 + 0.6(-100) & = -40 \\ x_3 = B & u_3 = r & C = 40 + 0.6(-100) & = -20 \\ & u_3 = l & C = 150 + (-100) & = 50 \end{array}$$

$$\rightarrow \quad \begin{array}{ll} J_3(R) = -40 & \mu_3(R) = m \\ J_3(B) = -20 & \mu_3(B) = r \end{array}$$

<u>Week 2</u>

$$\begin{array}{llll} x_2 = R & u_2 = n & C = 0 + 0.7(-20) + 0.3(-100-40) & = -56 \\ & u_2 = m & C = 20 + 0.4(-20) + 0.6(-140) & = -72 \\ x_2 = B & u_2 = r & C = 40 + 0.4(-20) + 0.6(-140) & = -52 \\ & u_2 = l & C = 150 - 100 - 40 & = 10 \end{array}$$

$$\rightarrow \qquad J_2(R) = -72 \qquad \mu_2(R) = m$$
$$J_2(B) = -52 \qquad \mu_2(B) = r$$

Week 1

$x_1 = R$	$u_1 = n$	$C = 0 + 0.7(-52) + 0.3(-100 - 72)$	$= -\,88$
	$u_1 = m$	$C = 20 + 0.4(-52) + 0.6(-172)$	$= -\,104$
$x_1 = B$	$u_1 = r$	$C = 40 + 0.4(-52) + 0.6(-172)$	$= -\,84$
	$u_1 = l$	$C = 150 - 100 - 72$	$= -\,22$

$$\rightarrow \qquad J_1(R) = -104 \qquad \mu_1(R) = m$$
$$J_1(B) = -84 \qquad \mu_1(B) = r$$

Week 0

$x_0 = R$ Machine is guaranteed to run in the 1^{st} week (since it is new)

$$\rightarrow \quad J_0(R) = -100 - 104 = -204 \quad , \quad \mu_0(R) = n$$

- Conclusion:
  - always maintain a running machine
  - always repair a broken machine
  - expected profit $\underline{-J_0(R) = 204}$

Problem 3 (Solution)

Proof. Apply the general DP algorithm to the given problem:

$$J_N(x_N) = \alpha^N g_N(x_N) \quad \Leftrightarrow \quad \underbrace{J_N(x_N) \cdot \alpha^{-N}}_{=:V_N(x_N)} = g_N(x_N)$$

$$J_k(x_k) = \min_{u_k \in U_k(x_k)} \mathop{\mathrm{E}}_{w_k} \Big(\alpha^k g_k(x_k, u_k, w_k) + J_{k+1}\big(f_k(x_k, u_k, w_k)\big)\Big)$$
$$\Leftrightarrow \quad J_k(x_k) \cdot \alpha^{-k} = \min_{u_k} \mathop{\mathrm{E}}_{w_k} \Big(g_k(x_k, u_k, w_k) + \alpha^{-k} J_{k+1}\big(f_k(x_k, u_k, w_k)\big)\Big)$$
$$\Leftrightarrow \quad \underbrace{J_k(x_k) \cdot \alpha^{-k}}_{V_k(x_k)} = \min_{u_k} \mathop{\mathrm{E}}_{w_k} \Big(g_k(x_k, u_k, w_k) + \alpha \cdot \underbrace{J_{k+1}\big(x_{k+1}\big) \cdot \alpha^{-(k+1)}}_{V_{k+1}(x_{k+1})}\Big)$$

In general, defining $V_k(x_k) := J_k(x_k) \cdot \alpha^{-k}$, yields

$$V_N(x_N) = g_N(x_N)$$
$$V_k(x_k) = \min_{u_k} \mathop{\mathrm{E}}_{w_k} \Big(g_k(x_k, u_k, w_k) + \alpha \cdot V_{k+1}\big(f_k(x_k, u_k, w_k)\big)\Big) \qquad \square$$

Problem 4 (Solution)

a) *Proof.* Definitions:

$$
\begin{aligned}
\pi^k &:= \{\mu_k, \mu_{k+1}, \ldots, \mu_{N-1}\} \\
J_k^*(x_k) &:= \min_{\pi^k} \underset{\substack{w_i \\ i=k,\ldots,N-1}}{\mathrm{E}} \left(\exp\left(g_N(x_N) + \sum_{i=k}^{N-1} g_i(x_i, \mu_i, w_i) \right) \right) \\
J_N^*(x_N) &:= \exp\bigl(g_N(x_N)\bigr)
\end{aligned}
$$

Show by induction that J_k^* are equal to J_k, i.e. for $k = 0$ we obtain desired result.

Start:

$$
k = N \quad \rightarrow \quad J_N^*(x_N) = \exp\bigl(g_N(x_N)\bigr) = J_N(x_N) \text{ (by definition)}
$$

Hypothesis: Assume, for k and all x_{k+1}, we have $J_{k+1}^*(x_{k+1}) = J_{k+1}(x_{k+1})$.

Step: Since $\pi^k = \{\mu_k,\ \pi^{k+1}\}$, $\forall x_k$

$$
\begin{aligned}
J_k^*(x_k) &= \min_{\pi^k} \underset{\substack{w_i \\ i=k,\ldots,N-1}}{\mathrm{E}} \left(\exp\Bigl(g_N(x_N) + \sum_{i=k+1}^{N-1} g_i(x_i, \mu_i, w_i) + g_k(x_k, \mu_k, w_k) \Bigr) \right) \\
&\Downarrow \text{ Principle of Optimality argument} \\
&= \min_{\mu^k} \underset{w_k}{\mathrm{E}} \Biggl(\exp\bigl(g_k(x_k, \mu_k, w_k)\bigr) \cdot \\
&\min_{\pi^{k+1}} \underset{\substack{w_i \\ i=k+1,\ldots,N-1}}{\mathrm{E}} \left(\exp\bigl(g_N(x_N) + \sum_{i=k+1}^{N-1} g_i(x_i, \mu_i, w_i) \bigr) \right) \Biggr) \\
&= \min_{\mu^k} \underset{w_k}{\mathrm{E}} \left(\exp\bigl(g_k(x_k, \mu_k, w_k)\bigr) \cdot J_{k+1}^*(x_{k+1}) \right) \\
&= \min_{u_k \in U_k} \underset{w_k}{\mathrm{E}} \left(J_{k+1}(x_{k+1}) \cdot \exp\bigl(g_k(x_k, u_k, w_k)\bigr) \right) = J_k(x_k)
\end{aligned}
$$

□

b) Definitions

$$
\begin{aligned}
g_k &= g_k(x_k, u_k) \\
V_k(x_k) &:= \ln J_k(x_k).
\end{aligned}
$$

Proof.

$$\begin{aligned}
V_N(x_N) &= \ln J_N(x_N) = \ln\Big(\exp\big(g_N(x_N)\big)\Big) = g_N(x_N) \\
V_k(x_k) &= \ln J_k(x_k) \\
&= \ln\left(\min_{u_k \in U_k} \mathop{\mathrm{E}}_{w_k}\Big(J_{k+1}(x_{k+1}) \cdot \exp\big(g_k(x_k, u_k)\big)\Big)\right) \\
&\Downarrow * \\
&= \min_{u_k \in U_k} \ln\left(\exp\big(g_k(x_k, u_k)\big) \mathop{\mathrm{E}}_{w_k}\Big(J_{k+1}(x_{k+1})\Big)\right) \\
&= \min_{u_k \in U_k}\left(g_k(x_k, u_k) + \ln\Big(\mathop{\mathrm{E}}_{w_k}\Big(J_{k+1}(x_{k+1})\Big)\Big)\right) \\
&= \min_{u_k \in U_k}\left(g_k(x_k, u_k) + \ln \mathop{\mathrm{E}}_{w_k}\Big(J_{k+1}\big(f_k(x_k, u_k, w_k)\big)\Big)\right) \qquad \square
\end{aligned}$$

$*$ Interchange of *ln* and *min* is admissible since *ln* is monotonically increasing for positive arguments.

Problem 5 (Solution)

Augment the state space by state $\overline{x}$:

$$\overline{x}_k = \begin{cases} 1 & \text{the process has not been terminated at } 0, \dots, k-1 \\ 0 & \text{otherwise} \end{cases}.$$

With this, the *system equation* reads:

$$\widetilde{x}_{k+1} := \begin{bmatrix} x_{k+1} \\ \overline{x}_{k+1} \end{bmatrix} = \begin{bmatrix} x_{k+1} = f_k(x_k, u_k, w_k) \\ \overline{x}_{k+1} = \begin{cases} 0 & \text{if } \overline{x}_k = 0 \lor u_k = \overline{u}_k \lor w_k = \overline{w}_k \\ 1 & \text{otherwise} \end{cases} \end{bmatrix},$$

where $\overline{u}_k$: termination decision

$\overline{w}_k$: termination disturbance.

The *cost function* reads:

$$\widetilde{g}_k(\widetilde{x}_k, u_k, w_k) = \begin{cases} g_k(x_k, u_k, w_k) + T & \text{if } (u_k = \overline{u}_k \lor w_k = \overline{w}_k) \land \overline{x}_k = 1 \\ \overline{x}_k \cdot g_k(x_k, u_k, w_k) & \text{otherwise} \end{cases}.$$

The *total cost* is

$$\sum_{k=0}^{N} \widetilde{g}_k(\widetilde{x}_k, u_k, w_k),$$

with $g_N(x_N, u_N, w_N) = g_N(x_N)$.

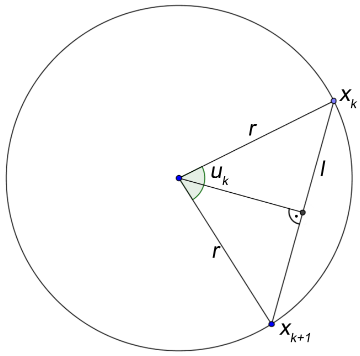
Problem 6 (Solution)

a) Formulate DP algorithm

- Let state x_k denote the angle on the circle specifying the location of the k-th point.
- Without loss of generality, we set $x_0 = 0$. Furthermore, $x_k \in [0, 2\pi) \quad \forall k = 1, \ldots, N$.
- Let u_k be the difference between x_{k+1} and x_k.
- Thus the update equation reads:

$$x_{k+1} = x_k + u_k, \text{with } \ x_0 = 0, \ x_N < 2\pi, \ u_k > 0 \quad \forall k = 1, \ldots, N-1. \tag{1}$$

- The length l of the line joining x_{k+1} and x_k is



$$\sin\left(\frac{u_k}{2}\right) = \frac{l}{2r} \quad \Leftrightarrow \quad l = 2r\sin\left(\frac{u_k}{2}\right),$$

where r denotes the radius of the circle.

- The length of the last segment joining x_N and x_0 is

$$2r\sin\left(\frac{2\pi - x_N}{2}\right) = 2r\sin\left(\pi - \frac{x_N}{2}\right) = 2r\sin\left(\frac{x_N}{2}\right),$$

where we used

$$\left.\begin{array}{r} \sin(x-\pi) = -\sin(x) \\ \sin(+x) = -\sin(-x) \end{array}\right\} \quad \sin(-x+\pi) = \sin(x). \tag{2}$$

- Defining

$$\begin{aligned} g_N(x_N) &:= 2r\sin\left(\frac{x_N}{2}\right) \\ g_k(u_k) &:= 2r\sin\left(\frac{u_k}{2}\right) \end{aligned}$$

the Dynamic Programming Problem is given by (1) and the objective to maximize the perimeter

$$\max_{\pi}\left[g_N(x_N) + \sum_{k=0}^{N-1} g_k(u_k)\right]. \tag{3}$$

b) Apply the DPA:

- Stage $k = N$:

$$J_N(x_N) = g_N(x_N) = 2r \sin\left(\frac{x_N}{2}\right)$$

- Stage $k = N-1$:

$$\begin{aligned} J_{N-1}(x_{N-1}) &= \max_{u_{N-1}} \left[g_{N-1}(u_{N-1}) + J_N(x_{N-1} + u_{N-1})\right] \\ &= 2r \max_{u_{N-1}>0} \left[\sin\left(\frac{u_{N-1}}{2}\right) + \sin\left(\frac{x_{N-1}+u_{N-1}}{2}\right)\right] \end{aligned}$$

differentiate with respect to u_{N-1} and set to zero:

$$\frac{1}{2}\cos\left(\frac{u_{N-1}}{2}\right) + \frac{1}{2}\cos\left(\frac{x_{N-1}+u_{N-1}}{2}\right) = 0$$

A sufficient condition for optimality is

$$\begin{aligned} \frac{u_{N-1}}{2} &= -\frac{x_{N-1}+u_{N-1}}{2} + \pi \\ u_{N-1} &= \underline{\pi - \frac{x_{N-1}}{2}} \end{aligned}$$

One can show graphically that this is indeed the maximum.

Plug this into $J_{N-1}(x_{N-1})$:

$$\begin{aligned} J_{N-1}(x_{N-1}) &= 2r\left[\sin\left(\frac{\pi}{2} - \frac{x_{N-1}}{4}\right) + \sin\left(\frac{x_{N-1}}{2} + \frac{\pi}{2} - \frac{x_{N-1}}{4}\right)\right] \\ &= \underline{4r \sin\left(\frac{\pi}{2} - \frac{x_{N-1}}{4}\right)} \end{aligned}$$

- Stage $k = N-2$:

$$\begin{aligned} J_{N-2}(x_{N-2}) &= \max\left[2r\sin\left(\frac{u_{N-2}}{2}\right) + J_{N-1}(x_{N-2}+u_{N-2})\right] \\ &= 2r \max\left[\sin\left(\frac{u_{N-2}}{2}\right) + 2\sin\left(\frac{\pi}{2} - \frac{x_{N-2}}{4} - \frac{u_{N-2}}{4}\right)\right] \end{aligned}$$

differentiate, set to 0:

$$\frac{1}{2}\cos\left(\frac{u_{N-2}}{2}\right) - \frac{1}{2}\cos\left(\frac{\pi}{2} - \frac{x_{N-2}}{4} - \frac{u_{N-2}}{4}\right) = 0$$

sufficient condition:

$$u_{N-2} = \underline{\frac{2\pi}{3} - \frac{x_{N-2}}{3}}$$

plug in into $J_{N-2}(x_{N-2})$:

$$\underline{J_{N-2}(x_{N-2}) = 6r\sin\left(\frac{\pi}{3} - \frac{x_{N-2}}{6}\right)}$$

- From the first two iterations, we *guess* the general form:

$$J_{N-k}(x_{N-k}) = 2(k+1)\cdot r \sin\left(\frac{\pi}{k+1} - \frac{x_{N-k}}{2(k+1)}\right) \tag{4}$$

$$u_{N-k} = \frac{2\pi}{k+1} - \frac{x_{N-k}}{k+1} \tag{5}$$

We prove by induction that this is indeed the solution.

Proof. (by induction)

- We have shown that (4), (5) are true for $k = 0, 1, 2$.
- Assume that (4), (5) are true for k; show that (4), (5) are true for $k+1$,

$$\begin{aligned}
&J_{N-k-1}(x_{N-k-1}) \\
&= \max_{u_{N-k-1}} \left[2r \sin\left(\frac{u_{N-(k+1)}}{2}\right) + J_{N-k}(x_{N-k}) \right] \\
&\Downarrow \text{ Induction hypothesis} \\
&= 2r \max_{u_{N-k-1}} \left[\sin\left(\frac{u_{N-k-1}}{2}\right) + (k+1)\sin\left(\frac{\pi}{k+1} - \frac{x_{N-k}}{2(k+1)}\right) \right] \\
&= 2r \max_{u_{N-k-1}} \left[\sin\left(\frac{u_{N-k-1}}{2}\right) + (k+1)\sin\left(\frac{\pi}{k+1} - \frac{x_{N-k-1} + u_{N-k-1}}{2(k+1)}\right) \right]
\end{aligned}$$

differentiate, set to 0, solve for u_{N-k-1}:

$$u_{N-k-1} = \frac{2\pi}{k+2} - \frac{x_{N-k-1}}{k+2}$$

plug into J_{N-k-1}:

$$J_{N-k-1}(x_{N-k-1}) = 2r(k+2)\sin\left(\frac{\pi}{k+2} - \frac{x_{N-k-1}}{2(k+2)}\right)$$ □

In particular, for $k = N$, we have

$$\begin{aligned}
J_{N-N}(x_{N-N}) = J_0(x_0) &= 2(N+1) \cdot r \cdot \sin\left(\frac{\pi}{N+1} - \frac{x_0}{2(N+1)}\right) \\
&= 2r(N+1)\sin\left(\frac{\pi}{N+1}\right),
\end{aligned}$$

which is the perimeter of a $(N+1)$-side polygon and

$$u_0 = \frac{2\pi}{N+1} \quad , \quad x_0 = 0.$$

It still needs to be shown that $u_k = u_0 = \frac{2\pi}{N+1}$ $\forall k = 0, \ldots, N-1$, i.e. all segments are the same length.

Conjecture:

$$x_k = \frac{k \cdot 2\pi}{N+1} \qquad u_k = \frac{2\pi}{N+1}$$

Proof. (by induction)

- Conjecture is true for $k = 0$.
- Assume true for some k, show that true for $k+1$:

$$\begin{aligned}
x_{k+1} &\overset{(1)}{=} x_k + u_k = \frac{k \cdot 2\pi}{N+1} + \frac{2\pi}{N+1} = \frac{(k+1) \cdot 2\pi}{N+1} \\
u_{k+1} &\overset{(5)}{=} \frac{2\pi}{(N-(k+1))+1} - \frac{x_{N-(N-(k+1))}}{(N-(k+1))+1} \\
&= \frac{2\pi}{N+1}
\end{aligned}$$ □

→ All side lengths are the same, $\frac{2\pi}{N+1}$, thus the $(N+1)$-side polygon is regular.