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Quiz 2 **December 21st, 2011**

Dynamic Programming & Optimal Control (151-0563-01)

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Solutions

Duration:	**45 minutes**
Number of Problems:	**1**
Permitted Aids:	None. Use only the prepared sheets for your solutions.

Problem 1 100%

Consider the system of water reservoirs shown in Figure 1

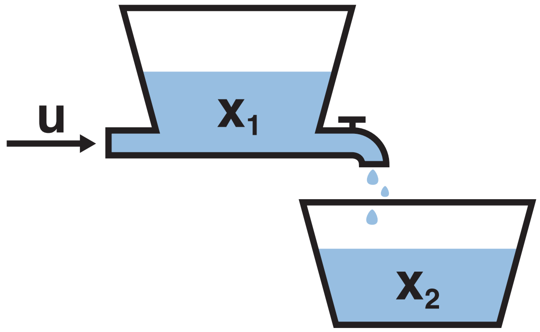

Figure 1: water reservoir system

where x_i, $i = 1, 2$ is the water volume of reservoir i and u is the flow rate of water into reservoir 1 via an external pump. Water exits reservoir 1 at a rate x_1 and enters reservoir 2.

a) Taking flow rate u as the control, find the system equations of the water reservoir system shown in Figure 1.

b) Assuming the external pump is unidirectional and has a maximum inflow rate of 2, i.e. $u \in U_b = [0, 2]$ compute the time optimal maneuver to fill each of the two empty reservoirs with 1 unit of water by applying the Minimum Principle. (When x_2 reaches 1 unit of water the valve between reservoir 1 and 2 will be closed to prevent more water flowing into reservoir 2.)

c) If the external pump is bidirectional and has a maximum flow rate of 2, i.e. $u \in U_c = [-2, 2]$, will the optimal maneuver to fill the two empty reservoirs with one unit of water be faster, slower or take the same amount of time as in the case with the unidirectional pump? Explain the reasons for your answer.

Solution 1

a)

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + u(t)\\ \dot{x}_2(t) &= x_1(t)\end{aligned}$$

b)

- The boundary conditions for the system are $x_1(0) = x_2(0) = 0$ and $x_1(T) = x_2(T) = 1$.

- The objective to minimize is
$$T = \int_0^T 1\ dt.$$
- The Hamiltonian is
$$H(x(t), u(t), p(t)) = 1 - p_1(t)x_1(t) + p_1(t)u(t) + p_2(t)x_1(t).$$
- Adjoint equations
$$\begin{aligned}&\dot{p}_2 = 0\ \Rightarrow p_2(t) = c\\ &\dot{p}_1 = p_1 - p_2 = p_1 - c\ \Rightarrow p_1(t) = \xi e^t + c \rightarrow \text{max. one zero crossing}\end{aligned}$$
- If $u^*(t)$is the optimal control and $x^*(t)$ is the optimal state trajectory, then the necessary condition for optimality is
$$u^*(t) = \underset{u \in U}{\operatorname{argmin}}\ H(x^*(t), u, p(t)).$$
- Since the Hamiltonian is linear in u, u will always be on a boundary of U:
$$u^*(t) = \begin{cases} 0 & \text{if } p_1(t) \geq 0\\ 2 & \text{if } p_1(t) < 0\end{cases}$$
- Since both containers are initially empty we start at $u = 2$. We know that we have at most one zero crossing of p_1. Therefore we will have to apply $u = 2$ until we have enough water in the reservoirs.
$$x_1(T_b) + x_2(T_b) = 2 = \int_0^{T_b} u(t)\ dt = \int_0^{t_{switch}} 2\ dt + \int_{t_{switch}}^{T_b} 0\ dt = 2t_{switch}$$
$\Rightarrow$ The optimal solution is to run the pump at $u = 2$ for 1 time unit and then switch it off and wait until enough water has run down into reservoir 2.

c) First of all, since the control set U_b is a subset of the control set U_c the maneuver cannot be slower because we can apply the solution obtained in b). To show that the maneuver using U_c is faster than the maneuver using U_b we can use a proof by contradiction:

Assume that the solution obtained in b) with U_b is also an optimal solution for c) with U_c. Then the solution from b) also needs to be a minimizer of the Hamiltonian given U_c. Since the adjoint equations and the Hamiltonian do not change, p_1 still has at most one zero crossing and the Hamiltonian is still linear in u. Therefore we know that the optimal solution is on the boundary of U_c and switches at most one time. But the solution obtained in b) is not on the boundary of U_c and therefore is not a minimizer for the Hamiltonian anymore. So the solution using U_c must be faster than the solution using U_b.