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Quiz 2 **December 9th, 2009**

Dynamic Programming & Optimal Control (151-0563-00)

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Solutions

Duration:	**45 minutes**
Number of Problems:	**2**
Permitted Aids:	None. Use only the prepared sheets for your solutions.

Problem 1 20 %

Please circle either 'Yes' or 'No' or 'Not enough information' in the following questions.

a) A given deterministic continuous-time optimal control problem is known to have an optimal solution. Furthermore, there exists a unique solution pair $\{(x(t),\, u(t)) \mid t \in [0,\, T]\}$ that satisfies the Minimum Principle.

Is the solution pair $\{(x(t),\, u(t)) \mid t \in [0,\, T]\}$ optimal?

Yes No Not enough information

b) For a given deterministic continuous-time optimal control problem, you find a solution to the Hamilton-Jacobi-Bellman Equation and a feedback law $\mu(t, x)$ which attains the minimum in the corresponding Hamilton–Jacobi-Bellman Equation.

Is the resulting feedback law $\mu(t, x)$ optimal?

Yes No Not enough information

Note: *For each question you get: +10 % for a correct answer, -5 % for an incorrect answer and 0 % for no answer. If you change your mind, please cross out all options ('Yes' and 'No' and 'Not enough information') and write either 'Yes' or 'No' or 'Not enough information' alongside, or leave it blank.*

Solution 1

a) Yes.

The Minimum Principle is a *necessary* condition for optimality; that is, not every solution pair $\{(x(t),\, u(t)) \mid t \in [0,\, T]\}$ that satisfies the Minimum Principle, has to be optimal; but, every optimal solution has to satisfy the Minimum Principle. In our case, we know that an optimal solution exists to the problem and there is only one solution pair satisfying the Minimum Principle. Therefore, the solution pair is optimal.

b) Yes.

The Hamilton-Jacobi-Bellman Equation is a *sufficient* condition; that is, if there is a solution to the Hamilton-Jacobi-Bellman Equation, it is optimal.

Problem 2 **80%**

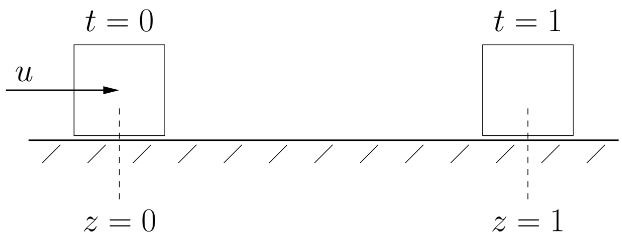

Figure 1

At time $t = 0$, a mass $m = 2$ is at rest at location $z = 0$. The mass is on a frictionless surface and it is desired to apply a force $u(t)$, $0 \leq t \leq 1$, such that at time $t = 1$, the mass is at location $z = 1$ with velocity $\dot{z} = 1$. In particular,

$$\ddot{z}(t) = \frac{1}{2}u(t), \quad 0 \leq t \leq 1, \tag{1}$$

with initial and terminal conditions:

$$\begin{aligned} z(0) &= 0, \quad \dot{z}(0) = 0, \\ z(1) &= 1, \quad \dot{z}(1) = 1. \end{aligned}$$

Of all the functions $u(t)$ that achieve the above objective, find the one that minimizes

$$\int_0^1 \rho u^2(t)dt,$$

where $\rho > 0$ is a given constant.

Solution 2

Introduce the state vector

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} z \\ \dot{z} \end{bmatrix}.$$

Using this notation, the system dynamics read

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \frac{1}{2}u(t) \end{bmatrix} = f\big(x(t), u(t)\big)$$

with initial and terminal conditions

$$x_1(0) = 0, \quad x_2(0) = 0,$$
$$x_1(1) = 1, \quad x_2(1) = 1.$$

Apply the Minimum Principle[1].

- The Hamiltonian is given by

$$\begin{aligned} H(x, u, p) &= g(x, u) + p^T f(x, u) \\ &= \rho u^2 + p_1 x_2 + \frac{1}{2} p_2 u. \end{aligned}$$

- The optimal input $u^*(t)$ is obtained by minimizing the Hamiltonian along the optimal trajectory. Differentiating the Hamiltonian with respect to u yields,

$$2\rho u + \frac{1}{2} p_2(t) = 0 \quad \Leftrightarrow \quad u = -\frac{1}{4\rho} p_2(t).$$

  Since the second derivative of H with respect to u is $2\rho > 0$, $u^*(t) = -\frac{1}{4\rho} p_2(t)$ is indeed the minimum.

- The adjoint equations,

$$\dot{p}_1(t) = 0$$
$$\dot{p}_2(t) = -p_1(t),$$

  are integrated and result in the following equations:

$$p_1(t) = \tilde{c}_1, \quad \tilde{c}_1 \text{ constant}$$
$$p_2(t) = -\tilde{c}_1 t - \tilde{c}_2, \quad \tilde{c}_2 \text{ constant}.$$

  Using this result, the optimal input is given by

$$u^*(t) = \frac{1}{4\rho}(\tilde{c}_1 t + \tilde{c}_2) = c_1 t + c_2,$$

  where the new constants $c_1 := \frac{\tilde{c}_1}{4\rho}$ and $c_2 := \frac{\tilde{c}_2}{4\rho}$ have been introduced for notational simplicity.

[1] Note that the terminal state is given in this problem. Therefore, there is no terminal condition on the co-states (as in the standard Minimum Principle), but the terminal condition on the states can be used instead to solve the set of differential equations.

- Using the initial and terminal conditions on x, we will solve for c_1 and c_2 next. For this purpose, we first need to solve for the state trajectories $x_1^*(t)$ and $x_2^*(t)$ using the system equation,

  $$\dot{x}_2^*(t) = \frac{1}{2}u^*(t) = \frac{1}{2}(c_1 t + c_2) \quad \Rightarrow \quad x_2^*(t) = \frac{1}{4}c_1 t^2 + \frac{1}{2}c_2 t + c_3, \quad c_3 \text{ constant,}$$

  and

  $$\dot{x}_1^*(t) = x_2^*(t) = \frac{1}{4}c_1 t^2 + \frac{1}{2}c_2 t + c_3 \quad \Rightarrow \quad x_1^*(t) = \frac{1}{12}c_1 t^3 + \frac{1}{4}c_2 t^2 + c_3 t + c_4, \quad c_4 \text{ constant.}$$

  Using the initial conditions $x_1(0) = 0$ and $x_2(0) = 0$, it follows that

  $$c_3 = c_4 = 0.$$

  Using this and the terminal conditions $x_1(1) = 1$ and $x_2(1) = 1$, we obtain

  $$\begin{aligned} x_1^*(t) &= \frac{1}{12}c_1 + \frac{1}{4}c_2 = 1 \\ x_2^*(1) &= \frac{1}{4}c_1 + \frac{1}{2}c_2 = 1. \end{aligned}$$

  Solving for c_1 and c_2 yields

  $$c_1 = -12, \quad c_2 = 8.$$

- Therefore, the optimal control is

  $$u^*(t) = -12t + 8,$$

  and the optimal state trajectory is

  $$\begin{aligned} x_1^*(t) &= z^*(t) = -t^3 + 2t^2 \\ x_2^*(t) &= \dot{z}^*(t) = -3t^2 + 4t. \end{aligned}$$