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Final Examination **January 28th, 2009**

Dynamic Programming & Optimal Control (151-0563-01) **Prof. R. D'Andrea**

Solutions

Exam Duration:	**150 minutes**
Number of Problems:	**4** (25% each)
Permitted Aids:	Textbook *Dynamic Programming and Optimal Control by Dimitri P. Bertsekas, Vol. I, 3rd edition, 2005, 558 pages.* Your written notes and class handouts. **No** calculators.
Important:	Use only these prepared sheets for your solutions.

General Hints:	$E(\cdot)$ denotes the expected value. $E(rs) = E(r)E(s)$, if two random variables r, s are independent.

Problem 1 **25%**

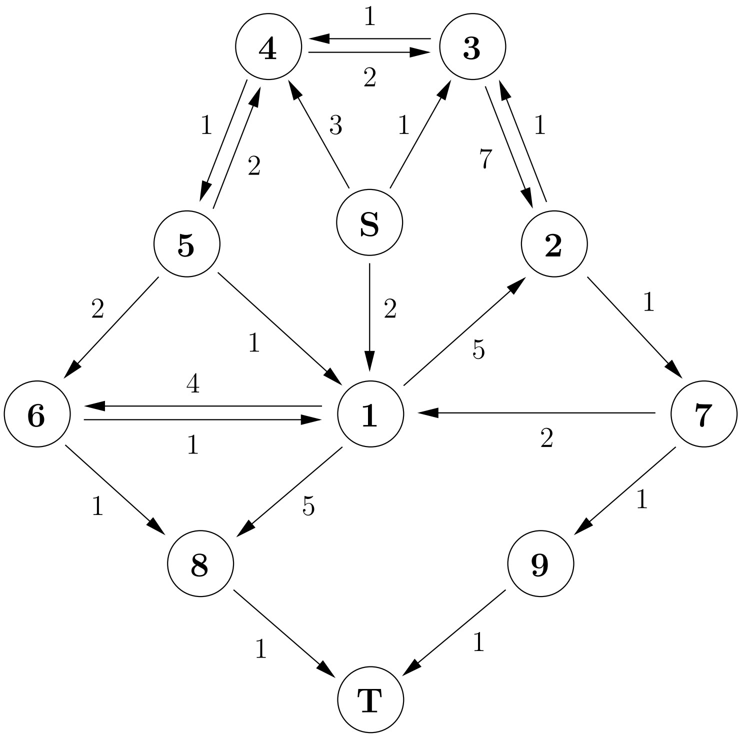

Figure 1

Find the shortest path from node S to node T for the graph given in Figure 1. Apply the *Label Correcting Method*. Use *Depth-First Search* to determine at each iteration which node to remove from `OPEN`; that is, the node is always removed from the top of `OPEN` and each node entering `OPEN` is placed at the top of `OPEN` (last-in/first-out policy).

Solve the problem by populating a table of the following form:

Iteration	Node exiting `OPEN`	`OPEN`	d_S	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_T
0	-	...											
1	S	...											
2	4	...											
...													

where the variable d_i denotes the length of the shortest path from node S to node i that has been found so far. Note that the first two nodes exiting `OPEN` are given.

State the resulting shortest path and its length.

Solution 1

Iteration	Node exiting OPEN	OPEN	d_S	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_T
0	-	S	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	S	4,3,1	0	2	∞	1	3	∞	∞	∞	∞	∞	∞
2	4	5,3,1	0	2	∞	1	3	4	∞	∞	∞	∞	∞
3	5	6,3,1	0	2	∞	1	3	4	6	∞	∞	∞	∞
4	6	8,3,1	0	2	∞	1	3	4	6	∞	7	∞	∞
5	8	3,1	0	2	∞	1	3	4	6	∞	7	∞	8
6	3	4,1	0	2	∞	1	2	4	6	∞	7	∞	8
7	4	5,1	0	2	∞	1	2	3	6	∞	7	∞	8
8	5	6,1	0	2	∞	1	2	3	5	∞	7	∞	8
9	6	8,1	0	2	∞	1	2	3	5	∞	6	∞	8
10	8	1	0	2	∞	1	2	3	5	∞	6	∞	7
11	1	-	0	2	∞	1	2	3	5	∞	6	∞	7

Alternative solution:

Iteration	Node exiting OPEN	OPEN	d_S	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_T
0	-	S	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	S	4,1,3	0	2	∞	1	3	∞	∞	∞	∞	∞	∞
2	4	5,1,3	0	2	∞	1	3	4	∞	∞	∞	∞	∞
3	5	6,1,3	0	2	∞	1	3	4	6	∞	∞	∞	∞
4	6	8,1,3	0	2	∞	1	3	4	6	∞	7	∞	∞
5	8	1,3	0	2	∞	1	3	4	6	∞	7	∞	8
6	1	2,3	0	2	7	1	3	4	6	∞	7	∞	8
7	2	3	0	2	7	1	3	4	6	∞	7	∞	8
8	3	4	0	2	7	1	2	4	6	∞	7	∞	8
9	4	5	0	2	7	1	2	3	6	∞	7	∞	8
10	5	6	0	2	7	1	2	3	5	∞	7	∞	8
11	6	8	0	2	7	1	2	3	5	∞	6	∞	8
12	8	-	0	2	7	1	2	3	5	∞	6	∞	7

The shortest path is $S \to 3 \to 4 \to 5 \to 6 \to 8 \to T$ with a total length of 7.

Problem 2 **25%**

Consider the following system

$$x_{k+1} = f_{i_k}(x_k, u_k, w_k)\,, \quad i_k = 1,\, 2\,,$$

where

$$f_1(x_k, u_k, w_k) = w_k x_k + w_k u_k$$
$$f_2(x_k, u_k, w_k) = w_k x_k - 2 w_k u_k\,.$$

The disturbance w_k takes the values 0 and 1 with equal probability. The input u_k is restricted to be 1 or -1.

State constraints are given as follows:

$$-2k \leq x_k \leq 2k\,.$$

Starting from an initial state $x_0 = 0$, the goal is to minimize the cost

$$\underset{w_0, w_1}{E}\left\{x_0^2 + x_1^2 + x_2^2\right\}.$$

The control inputs are u_k and i_k.

Apply the Dynamic Programming algorithm to find the optimal control policy and the optimal cost $J_0(0)$.

Solution 2

The optimal control problem is considered over a time horizon $N = 2$ and the cost, to be minimized, is defined by

$$g_2(x_2) = x_2^2 \qquad \text{and} \qquad g_k(x_k, u_k, w_k) = x_k^2, \quad k = 0, 1.$$

Note that the state x_k takes on only integer values since $w_k \in \{0,\, 1\}$, $u_k \in \{-1,\, 1\}$, and $x_0 = 0$.

The DP algorithm proceeds as follows:

2nd stage:
The recursion is started with

$$J_2(x_2) = x_2^2$$

for all feasible $x_2 \in \{-4,\, -3,\, \ldots,\, 2,\, 3,\, 4\}$.

1st stage:
Proceeding backwards, we get:

$$\begin{aligned} J_1(x_1) &= \min_{\substack{i_1 \in \{1,2\} \\ u_1 \in \{-1,1\}}} \underset{w_1}{E}\Big\{x_1^2 + J_2\big(f_{i_1}(x_1, u_1, w_1)\big)\Big\} \\ &= \min_{\substack{i_1 \in \{1,2\} \\ u_1 \in \{-1,1\}}} \underset{w_1}{E}\Big\{x_1^2 + \big(f_{i_1}(x_1, u_1, w_1)\big)^2\Big\}. \end{aligned}$$

The functions $f_1(x_k, u_k, w_k)$ and $f_2(x_k, u_k, w_k)$ are rewritten as

$$\begin{aligned} f_1(x_k, u_k, w_k) &= \ w_k\,(x_k + u_k) &&= w_k\; h_1(x_k, u_k) \\ f_2(x_k, u_k, w_k) &= \ w_k\,(x_k - 2u_k) &&= w_k\; h_2(x_k, u_k)\,. \end{aligned}$$

Using these definitions, we get

$$\begin{aligned} J_1(x_1) &= \min_{\substack{i_1 \in \{1,2\} \\ u_1 \in \{-1,1\}}} \underset{w_1}{E}\Big\{x_1^2 + w_1^2\big(h_{i_1}(x_1, u_1)\big)^2\Big\} \\ &= \min_{\substack{i_1 \in \{1,2\} \\ u_1 \in \{-1,1\}}} \Big\{x_1^2 + 0.5 \cdot 0^2 + 0.5 \cdot 1^2 \cdot \big(h_{i_1}(x_1, u_1)\big)^2\Big\} \\ &= x_1^2 \ + \ 0.5 \cdot \min_{\substack{i_1 \in \{1,2\} \\ u_1 \in \{-1,1\}}} \Big\{\big(h_{i_1}(x_1, u_1)\big)^2\Big\}. \end{aligned}$$

Now, by evaluating $h_{i_1}(x_1, u_1)$ for all feasible $x_1 \in \{-2,\, -1,\, 0,\, 1,\, 2\}$ considering possible input pairs,

$$(u_1, i_1) \in \Big\{(1, -1), (1, 1), (2, -1), (2, 1)\Big\},$$

the state-dependent minimizing input and the corresponding cost-to-go $J_1(x_1)$ are found:

$$
\begin{aligned}
J_1(-2) &= 4 + 0 = 4\,, && \text{with} \quad \mu^*(-2) = (2,-1)\\
J_1(-1) &= 1 + 0 = 1\,, && \text{with} \quad \mu_1^*(-1) = (1,1)\\
J_1(0) &= 0 + 0.5 = 0.5\,, && \text{with} \quad \mu_1^*(0) = (1,\pm 1)\\
J_1(1) &= 1 + 0 = 4\,, && \text{with} \quad \mu_1^*(1) = (1,-1)\\
J_1(2) &= 4 + 0 = 4\,, && \text{with} \quad \mu_1^*(2) = (2,1)\,.
\end{aligned}
$$

0th stage:

Finally with the initial condition $x_0 = 0$, the optimal cost is calculated by

$$
\begin{aligned}
J_0(0) &= \min_{\substack{i_0 \in \{1,2\}\\ u_0 \in \{-1,1\}}} \; \underset{w_0}{E}\Big\{0^2 + J_1\big(f_{i_0}\,(0,u_0,w_0)\big)\Big\}\\
&= \min_{\substack{i_0 \in \{1,2\}\\ u_0 \in \{-1,1\}}} \Big\{0.5 \cdot J_1\big(f_{i_0}\,(0,u_0,0)\big) + 0.5 \cdot J_1\big(f_{i_0}\,(0,u_0,1)\big)\Big\}\\
&= \min_{\substack{i_0 \in \{1,2\}\\ u_0 \in \{-1,1\}}} \Big\{0.5 \cdot J_1\big(0\big) + 0.5 \cdot J_1\big(f_{i_0}\,(0,u_0,1)\big)\Big\}\,.
\end{aligned}
$$

The input,

$$
\mu_0^*(0) = (1,\pm 1)\,,
$$

minimizes the cost function and results in

$$
\begin{aligned}
J_0(0) &= 0.5 \cdot J_1\big(0\big) + 0.5 \cdot J_1\big(\pm 1\big)\\
&= 0.25 + 0.5 = 0.75\,.
\end{aligned}
$$

Problem 3 25%

Consider the following discrete-time system:

$$\begin{aligned} x_{k+1} &= x_k + u_k + w_k \,, \qquad k = 0, 1, \ldots, \infty \\ y_k &= x_k \,, \end{aligned}$$

where x_k, u_k, w_k, and y_k are real numbers. The initial condition is $x_0 = 1$. The w_k are independent random numbers with $E(w_k) = 0$ and $E(w_k^2) = 1$.
The cost function is the following:

$$J = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} E\left(\frac{x_k^2}{2} + u_k^2 \right) .$$

We restrict ourselves to the following class of feedback laws:

$$u_k = F y_k \,,$$

where F is a constant gain.

a) Find the gain F that minimizes the cost J.

b) Now consider the measurement corrupted by noise:

$$y_k = x_k + v_k \,,$$

where the v_k are independent random numbers with $E(v_k) = 0$ and $E(v_k^2) = 1$. What is the cost J if the **same** feedback gain that you found in part a) is used?

Solution 3

a) This is an infinite horizon, perfect information problem for a linear system with a quadratic cost function. The optimal feedback strategy is a constant gain, which coincides with the class of feedback laws that we are considering in this problem. We therefore just have to find the optimal LQR gain for

$$A = 1, \quad B = 1, \quad Q = \frac{1}{2}, \quad \text{and} \quad R = 1.$$

Solve the Riccati equation:

$$K = K - \frac{K^2}{K+1} + \frac{1}{2} \quad \Leftrightarrow \quad \frac{K^2}{K+1} = \frac{1}{2}.$$

The positive solution therefore is $K = 1$. The optimal feedback gain is thus

$$F = \frac{-K}{K+1} = -\frac{1}{2}.$$

b) Consider the closed loop with $u_k = -\frac{1}{2}y_k = -\frac{1}{2}x_k - \frac{1}{2}v_k$:

$$x_{k+1} = x_k - \frac{1}{2}x_k - \frac{1}{2}v_k + w_k = \frac{1}{2}x_k - \frac{1}{2}v_k + w_k. \tag{1}$$

The cost becomes

$$\begin{aligned}
J &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \left(E\left(\frac{x_k^2}{2} + u_k^2 \right) \right) \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \left(\frac{1}{2}E(x_k^2) + E\left(\frac{1}{4}x_k^2 + \frac{1}{2}x_k v_k + \frac{1}{4}v_k^2 \right) \right) \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \left(\frac{1}{2}E(x_k^2) + \frac{1}{4}E(x_k^2) + \frac{1}{4}E(v_k^2) \right) \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \left(\frac{3}{4}E(x_k^2) + \frac{1}{4} \right),
\end{aligned}$$

where we used $E(x_k v_k) = E(x_k)E(v_k) = 0$, which holds because v_k is an independent random variable and $E(v_k) = 0$.

Now, using (1) and by independence of v_k and w_k, we have

$$\begin{aligned}
E(x_{k+1}^2) &= E\left(\frac{1}{4}x_k^2 + \frac{1}{4}v_k^2 + w_k^2 - \frac{1}{2}x_k v_k + x_k w_k - v_k w_k \right) \\
&= \frac{1}{4}E(x_k^2) + \frac{1}{4}E(v_k^2) + E(w_k^2) \\
&= \frac{1}{4}E(x_k^2) + \frac{1}{4} + 1 \\
&= \frac{1}{4}E(x_k^2) + \frac{5}{4}.
\end{aligned} \tag{2}$$

Let $\alpha = \lim_{k\to\infty} E(x_k^2)$, which exists since the recursion (1) is stable. Substituting this in (2), we can solve for α,

$$\alpha = \frac{1}{4}\alpha + \frac{5}{4} \quad \Leftrightarrow \quad \alpha = \frac{5}{3}.$$

Therefore, the cost is

$$
\begin{aligned}
J &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \left(\frac{3}{4} \cdot \frac{5}{3} + \frac{1}{4} \right) \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \left(\frac{3}{2} \right) \\
&= \frac{3}{2}.
\end{aligned}
$$

Problem 4 **25%**

Consider the following nonlinear differential equations

$$\dot{x}_1(t) = x_2(t)u_1(t)$$
$$\dot{x}_2(t) = x_1(t)u_2(t)$$

with initial conditions

$$x_1(0) = -1$$
$$x_2(0) = 0\,,$$

where all variables are real numbers.
The control inputs have the following constraints:

$$|u_1(t)| \leq 1 \quad \text{and} \quad |u_2(t)| \leq 1.$$

a) Find control inputs $u_1(t)$, $u_2(t)$ which drive the system to $(x_1, x_2) = (1, 0)$ as quickly as possible.

b) Does it take longer or shorter to drive the system to $(x_1, x_2) = (0, 0)$?

Solution 4

a) The goal is to find an input trajectory $(u_1(t), u_2(t)),\ t \in [0, T]$ which accomplishes the transfer from

$$(x_1(0), x_2(0)) = (-1, 0) \quad \text{to} \quad (x_1(T), x_2(T)) = (1, 0)$$

in *minimum time*. Thus, we want to

$$\text{minimize} \;\; T = \int_0^T 1\, dt\,.$$

Using the standard notation of the cost functional,

$$h(x(T)) + \int_0^T g(x(t), u(t))dt\,,$$

that is

$$g(x) = 1 \quad \text{and} \quad h(x(T)) = 0\,.$$

Apply the Minimum Principle.

- The Hamiltonian is given by

$$\begin{aligned} H(x, u, p) &= g(x, u) + p^T f(x, u) \\ &= 1 + p_1 x_2 u_1 + p_2 x_1 u_2\,. \end{aligned}$$

- The adjoint equations follow from the equation above:

$$\begin{aligned} \dot{p}_1(t) &= -\frac{\partial H}{\partial x_1}\left(x^*(t), u^*(t), p(t)\right) = -p_2(t) u_2^*(t) \\ \dot{p}_2(t) &= -\frac{\partial H}{\partial x_2}\left(x^*(t), u^*(t), p(t)\right) = -p_1(t) u_1^*(t)\,. \end{aligned}$$

  Since we have a Fixed Terminal State Problem, there are no boundary conditions on p.

- In addition, for optimal time problems, the Hamiltonian satisfies

$$H\left(x^*(t), u^*(t), p(t)\right) = 1 + p_1(t) x_2^*(t) u_1^*(t) + p_2(t) x_1^*(t) u_2^*(t) = 0 \tag{3}$$

  for all $t \in [0, T]$.

- The optimal input $u^*(t)$ is obtained by minimizing the Hamiltonian along the optimal trajectory

$$u^*(t) = \underset{\substack{|u_1| \le 1 \\ |u_2| \le 1}}{\arg\min} \left\{ H\left(x^*(t), u, p(t)\right) \right\} = \underset{\substack{|u_1| \le 1 \\ |u_2| \le 1}}{\arg\min} \left\{ 1 + p_1 x_2^* u_1 + p_2 x_1^* u_2 \right\}$$

  Since the Hamiltonian is linear in u_1 and u_2, it follows that the optimal strategy is bang-bang; that is,

$$u_1^*(t) = \begin{cases} -1, & \text{if } p_1 x_2^* \ge 0 \\ 1, & \text{if } p_1 x_2^* < 0 \end{cases} \tag{4}$$

and

$$u_2^*(t) = \begin{cases} -1, & \text{if } p_2 x_1^* \geq 0 \\ 1, & \text{if } p_2 x_1^* < 0\,, \end{cases} \tag{5}$$

assuming that there are no intervals $\mathcal{T} = [t_1, t_2]\,,\ 0 \leq t_1 < t_2 \leq T,$ where

$$p_1(t) x_2^*(t) = 0 \quad \text{or} \quad p_2(t) x_1^*(t) = 0, \qquad \forall\, t \in \mathcal{T}\,. \tag{6}$$

Prove bang-bang strategy.

In order to have a bang-bang time-optimal solution, we have to show that assumption (6) holds.

First, assume $x_1^*(t) \equiv 0 \quad \forall t \in \mathcal{T}$. Then,

$$\dot{x}_2^*(t) = x_1^*(t) u_2^*(t) \equiv 0 \quad \forall t \in \mathcal{T}.$$

It follows that

$$x_2^*(t) \equiv const \quad \forall t \in \mathcal{T}.$$

That is, we do not move anywhere! Since the differential equations for x_1 and x_2 are symmetric, we can proceed in the same way for $x_2^*(t) \equiv 0 \quad \forall t \in \mathcal{T}$. So, the case, where x_1 or x_2 are 0 for any interval $\mathcal{T}$, is not interesting, since nothing happens. We do not move.

Now, assume $p_1(t) \equiv 0 \quad \forall t \in \mathcal{T}$. Then,

$$\dot{p}_1(t) = -p_2(t) u_2^*(t) \stackrel{!}{=} 0 \quad \forall t \in \mathcal{T}.$$

In order to satisfy the equation above, we have two options:

a) $p_2(t) \equiv 0 \quad \forall t \in \mathcal{T}$; however, this is a contradiction to (3).

b) $u_2^*(t) \equiv 0 \quad \forall t \in \mathcal{T}$; but then, (3) does not hold either.

Because of symmetry, we can do the same for $p_2(t) \equiv 0 \quad \forall t \in \mathcal{T}$.

To sum up, above's considerations show that the optimal input $u^*(t)$ has to be bang-bang.

Consider zero-switch solution.

We first consider the case, where $u_1^*(t) = const$ and $u_2^*(t) = const$ over the whole time horizon $0 \leq t \leq T$.

Distinguish two different cases:

I) $\mathbf{u_1^* = u_2^* = u = \pm 1}$:

In this case,

$$\dot{x}_1^* = x_2^* u$$
$$\dot{x}_2^* = x_1^* u\,.$$

That is,

$$\ddot{x}_1^* = u^2 x_1^* = x_1^* \qquad \text{and} \tag{7}$$

$$x_2^* = \frac{\dot{x}_1^*}{u}\,. \tag{8}$$

It follows that,

$$x_1^*(t) = A\cosh t + B\sinh t$$
$$x_2^*(t) = \frac{A}{u}\sinh t + \frac{B}{u}\cosh t\,,$$

where A, B are constant and cosh and sinh denote the hyperbolic cosines and sines:

$$\cosh t := \frac{e^t + e^{-t}}{2} \quad \text{and} \quad \cosh t := \frac{e^t - e^{-t}}{2}\,.$$

The constants A and B are chosen, such that the initial conditions are satisfied:

$$x_1^*(0) = A = -1$$
$$x_2^*(0) = \frac{B}{u} = 0\,.$$

Finally, we get

$$x_1^*(t) = -\cosh t \quad \text{and} \quad x_2^*(t) = -\frac{1}{u}\sinh t\,.$$

However, this solution does not solve our problem. It never reaches our final destination $(x_1, x_2) = (1, 0)$; that is, $x_1^*(t) \neq 1$ and $x_2^*(t) \neq 0$ for any $t > 0$.

II) $\mathbf{u_1 = -u_2 = u = \pm 1}$:
In this case,

$$\dot{x}_1^* = x_2^* u$$
$$\dot{x}_2^* = -x_1^* u\,.$$

That is,

$$\ddot{x}_1^* = -u^2 x_1^* = -x_1^* \qquad \text{and}$$
$$x_2^* = \frac{\dot{x}_1^*}{u}\,.$$

It follows that

$$x_1^*(t) = A\cos t + B\sin t$$
$$x_2^*(t) = -\frac{A}{u}\sin t + \frac{B}{u}\cos t$$

with A, B being constant. From the initial conditions, we determine

$$x_1^*(0) = A = -1$$
$$x_2^*(0) = \frac{B}{u} = 0\,.$$

Finally, we get

$$x_1^*(t) = -\cos t \quad \text{and} \quad x_2^*(t) = \frac{1}{u}\sin t\,. \tag{9}$$

Note that for $T = \pi$, the final state is reached, $x_1^*(T) = 1$ and $x_2^*(T) = 0$.
What still has to be checked, is that (9) and the corresponding costates $p_1(t)$ and

$p_2(t)$ satisfy the necessary conditions of the Minimum Principle, namely Equations (5), (3), and (4). In order to verify these conditions, we have to calculate the costates:

$$\begin{aligned}\dot{p}_1 &= p_2 u \\ \dot{p}_2 &= -p_1 u\,.\end{aligned}$$

That is,

$$\begin{aligned}\ddot{p}_1 &= -u^2 p_1 = -p_1 \\ \dot{p}_2 &= \frac{\dot{p}_1}{u}\,.\end{aligned}$$

It follows that

$$p_1(t) = C\cos t + D\sin t \tag{10}$$

$$p_2(t) = -\frac{C}{u}\sin t + \frac{D}{u}\cos t\,. \tag{11}$$

With these results, Equations (10) and (11), and the state trajectory (9), Equation (3) reads as

$$\begin{aligned}H\left(x^*(t), u^*(t), p(t)\right) &= 1 + p_1(t)x_2^*(t)\text{ï?‘}\frac{1}{2}, u - p_2(t)x_1^*(t)\,u \\ &= 1 + C\cos t\sin t + D\sin^2 t - C\cos t\sin t + D\cos^2 t \\ &= 1 + D \\ &\overset{!}{=} 0\,.\end{aligned}$$

That is $D = -1$. Moreover, in order to satisfy Equations (4) and (5), the constant C has to be chosen: $C = 0$.

To conclude, we found a trajectory (9) transfering the state from

$$\big(x_1(0), x_2(0)\big) = (-1, 0) \quad \text{to} \quad \big(x_1(T), x_2(T)\big) = (1, 0)\,, \quad \text{in time } T = \pi\,,$$

and satisfying the necessary conditions of the Minimum Principle. The trajectory describes a circle which, depending on the choice of $u \in \{1, -1\}$, lies above or under the x_1-axis.

Important to note is that the Minimum Principle is just a necessary condition. There might be a multiple-switch solution which is faster; i.e., with $T < \pi$. Proving that there is actually no other trajectory satisfying the necessary conditions is quite cumbersome and not required here. Nevertheless, intuition tells us that for a fast transition from the initial to the final state, $\dot{x}_1(t) = x_2(t)u_1(t)$ should be chosen large. Therefore, for an optimal solution, $|x_2(t)|$ should increase quickly during the first half of the trajectory and decrease at the end. This behavior can be observed when carefully looking at (9).

⇒ **Grading: Required for full grade was the derivation of the zero switch solution II), which results in a circular trajectory from the initial state to the final destination, and the verification of the necessary conditions for this trajectory.**

b) Longer. In fact, it takes an infinite amount of time to drive the system to $(0,0)$. Why? Assume that we can drive the system to $(0,0)$ in a finite time T, $T < \infty$. Define

$$\begin{aligned} z_1(t) &= x_1(T-t), & v_1(t) &= u_1(T-t), \\ z_2(t) &= x_2(T-t), & v_2(t) &= u_2(T-t). \end{aligned}$$

Then,

$$\begin{aligned} \dot{z}_1(t) &= -\dot{x}_1(T-t) = -x_2(T-t)\,u_1(T-t) = -z_2(t)\,v_1(t) \\ \dot{z}_2(t) &= -\dot{x}_2(T-t) = -x_1(T-t)\,u_2(T-t) = -z_1(t)\,v_2(t) \end{aligned}$$

with initial conditions

$$\begin{aligned} z_1(0) &= 0 \\ z_2(0) &= 0\,. \end{aligned}$$

The inputs u_1 and u_2 are bounded and, therefore, v_1 and v_2 are bounded. From the initial conditions, it follows

$$\dot{z}_1(t) = \dot{z}_2(t) = 0 \quad \forall\, t > 0\,,$$

which is a contradiction to $z_1(T) = -1$, our initial condition in the original problem definition.