Dynamic Programming and Optimal Control

Recitation #2

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$$J_N(x_N) = g_N(x_N)$$

$$J_k(x_k) = \min_{u_k \in U_k(x_k)} \mathop{E}_{\omega_k} \left\{ g_k(x_k, u_k, \omega_k) + J_{k+1}\left(f_k(x_k, u_k, \omega_k)\right) \right\}$$

Refresher: Proof by Induction

Theorem

$1 + 2 + \ldots + n = \frac{n(n+1)}{2}, \quad n \in \mathbb{N}$

Proof.

- True for $n = 1$
- Assume $n = k$ holds: $1 + 2 + ... + k = \frac{k(k+1)}{2}$ (Induction Hypothesis)
- Show $n = k + 1$ holds:

$$\begin{aligned}
1 + 2 + ... + k + (k+1) &= k(k+1)/2 + (k+1) \\
&\quad \text{(by the Induction Hypothesis)} \\
&= (k(k+1) + 2(k+1))/2 \\
&= (k+2)(k+1)/2 \\
&= (k+1)(k+2)/2
\end{aligned}$$

①. Revising proof by induction

②. Revising Dynamic Programming Algorithm.

③. Exercise 1.3 pg 52 (solution in problem set)

[for a similar question see Problem 2, Quiz 1, 2011]

1.3

State space $S_k = \{R, B\}$: R — machine running
B — machine broken

control space $U_k(x_k)$

$u_k(R) = \{n, m\}$ n — no maintenance, m — maintenance

$u_k(B) = \{r, l\}$ r — repair, l — replace

System dynamics

$$x_{k+1} = \omega_k$$

where

$$P(\omega_k = B \mid x_k = R, u_k = m) = 0.4$$
$$P(\omega_k = R \mid x_k = R, u_k = m) = 1 - 0.4 = 0.6$$
$$P(\omega_k = B \mid x_k = R, u_k = n) = 0.7$$
$$P(\omega_k = R \mid x_k = R, u_k = n) = 1 - 0.7 = 0.3$$
$$P(\omega_k = B \mid x_k = B, u_k = r) = 0.4$$
$$P(\omega_k = R \mid x_k = B, u_k = r) = 1 - 0.4 = 0.6$$
$$P(\omega_k = R \mid x_k = B, u_k = l) = 1$$
$$P(\omega_k = B \mid x_k = B, u_k = l) = 1 - 1 = 0$$

stage cost = input cost C_u + Gain g_k

$C_n = 0$

$C_m = 20$ (\$)

$C_r = 40$

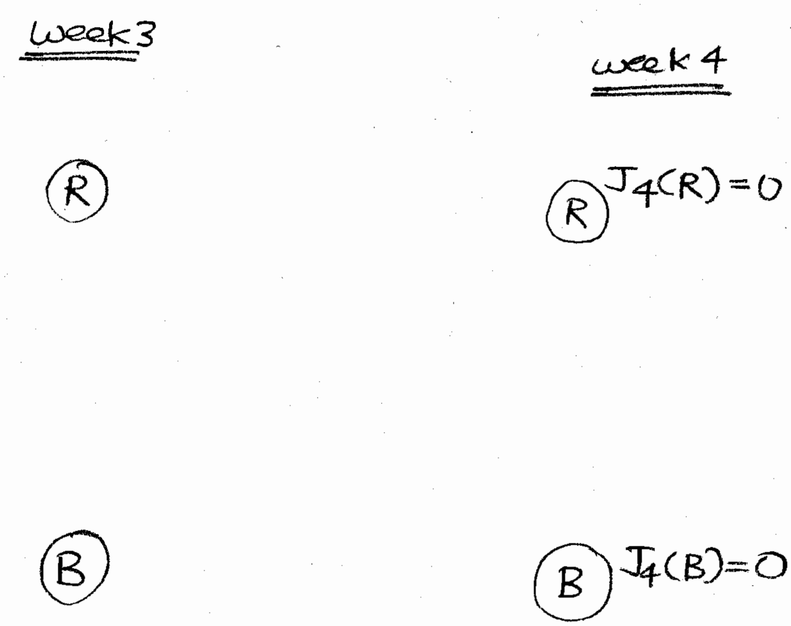
$C_l = 150$

$$g_k = \begin{cases} -G & \text{if } \omega_k = R \\ 0 & \text{if } \omega_k = B \end{cases}$$

Terminal cost $g_N(R) = g_N(B) = 0$

$J_4(B) = 0$

$\underline{J_3(\cdot)}$



Let's calculate $J_3(R)$ in detail

$$J_3(R) = \min_{\substack{u_3 \in U_3(R) \\ = \{n, m\}}} E\left\{ \underbrace{\text{stage cost}}_{g_3(x_3, u_3, \omega_3)} + J_4(\overbrace{\omega_3}^{x_4 \,=\, \omega_3}) \right\}$$

Using $J_4(\omega_3) = 0 \quad \forall \omega_3 \in \{R, B\}$

$$J_3(R) = \min \left\{ \underset{\omega_3}{E}\left[\text{stage cost when } \begin{matrix} x_3 = R \\ u_3 = n \end{matrix} + J_4(\omega_3) \right], \right.$$

$$\left. \underset{\omega_k}{E}\left[\text{stage cost when } \begin{matrix} x_3 = R \\ u_3 = m \end{matrix} + J_4(\omega_3) \right] \right\}$$

$$= \min \left\{ \begin{matrix} P(\omega_3 = R \mid x_3 = R, u_3 = n)(-100 + 0) + \\ P(\omega_3 = B \mid x_3 = R, u_3 = n)(10) \end{matrix}, \right.$$

$$\left. \begin{matrix} P(\omega_3 = R \mid x_3 = R, u_3 = m)(+20 - 100 + 0) \\ P(\omega_3 = B \mid x_3 = R, u_3 = m)(+20) \end{matrix} \right\}$$

$$= \min \{ 0.3(-100), \; 0.6(-80) + 0.4(+20) \} = \min \{-30, -40$$

$$\boxed{J_3(R) = -40} \qquad \boxed{u_3(R) = m}$$

$$\min_{\substack{u_3 \in U_3(B) \\ = \{r, h\}}} E_{\omega_3} \left[\text{stage cost} + J_4(\omega_4) \right]$$

$$= \min \left\{ \begin{array}{l} P(\omega_3 = R \mid x_3 = B, u_3 = r)(+40 - 100 + 0) + \\ \quad P(\omega_3 = B \mid x_3 = B, u_3 = r)(+40 + 0), \\ P(\omega_3 = R \mid x_3 = B, u_3 = h)(+150 - 100 + 0) + \\ \quad P(\omega_3 = B \mid x_3 = B, u_3 = r)(+150) \end{array} \right\}$$

$$= \min \left\{ 0.6(-60) + 0.4(+40),\ 1(+50) + 0(+150) \right\}$$

$$= \min \{ -20, +50 \}$$

$$\boxed{J_3(B) = -20}$$

$$\Rightarrow \boxed{\mu_3(B) = r}$$

Similarly

week 2

$x_2 = R$ $\quad u_2 = n$ $\quad C = 0 + 0.7(-20) + 0.3(-100 - 40) = -56$

$\quad\quad\quad\;\; u_2 = m$ $\quad C = 20 + 0.4(-20) + 0.6(-100 - 40) = -72$

for the rest see

A graphical way